Plane Partitions in Number Theory and Algebra

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November 20, 2017

Abstract

In this thesis we use classical and modern mathematical techniques to compute generating functions for plane partitions. We begin by introducing integer partitions and related generating functions, which follows from early works by Leonard Euler in the 1740s. The concept of partitions is then generalised to plane partitions. We will derive generating functions, as sums over Schur functions, for particular types of plane partitions. We then explicitly evaluate some of these sums using methods from representation theory, which follows from the work of Ian G. Macdonald in 1979. Thereafter, we examine plane partitions using modern mathematical methods resulting from problems in theoretical physics. In particular we follow the works of Andrei Y. Okounkov, Nicolai Y. Reshetikhin and Cumrun Vafa, in the 2000s, by studying vertex operators on the fermionic Fock space, which provide powerful techniques for enumerating plane partitions.

Acknowledgements

I foremost wish to thank my primary supervisor Prof. Ole Warnaar. I greatly appreciate his continual support throughout my honours year and for always making the time to assist me with my thesis. I also thank my secondary supervisor Dr. Chul-hee Lee for providing helpful feedback on my thesis.

Contents

Introduction

The theory of partitions has a long and rich history spanning hundreds of years, and has led to many advancements in mathematics. One of the first recorded accounts of the theory of partitions is in a letter Gottfried W. Leibniz wrote to Johann Bernoulli in 1674, [1]. In this letter Leibniz asked about the number of partitions of an integer, having observed there being three partitions of 3 namely 3, $2+1$ and $1+1+1$. It is easy for one to write down all partitions of small non-negative integers. For example, one could easily check that there are five partitions of 4 and seven partitions of 5. However, this will quickly become overwhelming for seemingly small non-negative integers. For example, there are over two-hundred thousand partitions of 50. Therefore, counting partitions is infeasible, especially in a time before computers. This led to the question: is there a function $p(n)$ that gives the number of partitions of a non-negative integer n? This question led to many great mathematical advancements by the likes of Leonhard Euler, James J. Sylvester, Percy A. MacMahon, Godfrey H. Hardy, Srinivasa Ramanujan and Hans Rademacher. A partial solution to the partition function $p(n)$ was given by Hardy and Ramanujan in 1918, [2]. In 1937 Rademacher refined the work of Hardy and Ramanujan into an exact formula for the partition function [3]. Despite this formula being very complicated, the generating function for $p(n)$ is relatively simple. It was first given by Leonhard Euler in the 1740s. We discuss this generating function in Chapter 1. We will also discuss partitions with particular restrictions, such as bounded partitions. These too have nice generating functions which we shall also derive in Chapter 1.

In Chapter 1 we also introduce plane partitions. These are a generalisation of partitions, and were originally introduced by Percy A. MacMahon in 1895. Similar to partitions we may like to know the number of plane partitions of a non-negative integer n. As a partition is a special type of plane partition we know that the answer to this will be at least the number of partitions of n. For example, there are six plane partitions of 3. Like the partition function, it may be better to first ask: what is the generating function for plane partitions? Moreover, we may like to know the generating function for particular types of plane partitions, such as bounded plane partitions. These questions require more advanced mathematical techniques than what is required to derive the generating functions related to ordinary partitions. In Chapter 1 we provide the generating functions for bounded, column strict and symmetric plane partitions. We express these as sums over Schur functions, which are a class of symmetric functions introduced by Augustin-Louis Cauchy in 1815, [4].

In 1898 MacMahon conjectured a product form for the generating function of bounded symmetric plane partitions, although he was not able to prove this. It would not be until eighty years later that MacMahon's conjecture was proved true. It was proved independently, by very different means, by George E. Andrews [5] [6] in 1978 and Ian G. Macdonald [7] in 1979. Macdonald's proof was given in his famous and highly influential book titled "Symmetric Functions and Hall Polynomials". It was representation theory that allowed Macdonald to discover a proof of MacMahon's conjecture. In Chapter 2 we will give an introduction to root systems and give an outline of Macdonald's proof.

Having discussed classical results of plane partitions in Chapters 1 and 2, in Chapter 3 we will move to discussing modern mathematical techniques which provide powerful methods for studying plane partitions. Namely, we will examine vertex operators on the fermionic Fock space. Not only can these vertex operators be used to prove classical results, they can also be used to study plane partitions with certain properties which may have been impossible to do with the classical techniques of MacMahon and Macdonald. In particular using these vertex operators we can study plane partitions with certain asymptotic properties. This is of interest in theoretical physics as it is associated with Calabi–Yau manifolds. This led to many advancements in this field of research. In Chapter 3 we will discuss and prove results from the theory of vertex operators on the fermionic Fock space. Following that we will use ideas from Andrei Y. Okounkov, Nicolai Y. Reshetikhin and Cumrun Vafa [8] [9] [10] to calculate a product form for the generating function for plane partitions.

Chapter 1

Integer and Plane Partitions

In this Chapter we introduce integer and plane partitions, and discuss related generating functions. In Section 1.1 we formally define integer partitions and important related constructions, such as Young diagrams. We also derive product forms for the generating functions of bounded and unbounded partitions. In Section 1.2 we generalise the idea of integer partitions into plane partitions. In Section 1.3 we calculate generating functions for bounded, column strict and symmetric plane partitions as sums over Schur functions. Thereafter, in Section 1.4 we generalise the Schur functions and prove some results that will be needed in Chapter 3.

1.1 Integer Partitions

A (integer) *partition* λ is a sequence $(\lambda_1, \lambda_2, \lambda_3, \dots)$ of weakly decreasing non-negative integers such that only finitely many of them are non-zero. We say that λ is a *partition of* n if $n = \sum_{j=1}^{\infty} \lambda_j$. This is denoted by $\lambda \vdash n$ or $|\lambda| = n$. The non-zero λ_j are called the *parts* of λ. The number of parts of λ is denoted by $\ell(\lambda)$ and is called the *length* of λ. We are often not interested in the string of zeros in a partition. Hence, if λ is a partition of length n then we may write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$. For example $(6, 3, 2, 0, 0, \dots) = (6, 3, 2)$ is a partition of 11 of length 3. For partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ and $\mu = (\mu_1, \mu_2, \dots)$ we say that μ is *contained in* λ, denoted $\mu \subseteq \lambda$, if $\mu_i \le \lambda_i$ for all i. Moreover, if $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots$ then we write $\mu \prec \lambda$ and say that λ and μ are *interlacing* partitions.

A partition $\lambda = (\lambda_1, \lambda_2, \dots)$ may be represented diagrammatically by its corresponding *Ferrers graph*, which is given by descending rows of points where the first row contains λ_1 points, the second row contains λ_2 points, etc., (all left aligned). For example the partition $(6, 3, 3, 1)$ corresponds to the Ferrers graph:

$$\begin{array}{llllll}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & & & \\
\bullet & \bullet & \bullet & & & \\
\bullet & & & & &
\end{array}$$

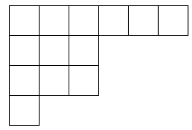
This graphical representation of partitions is associated to Norman M. Ferrers through a 1853 paper by James J. Sylvester [11] where he describes Ferrers' use of these graphs to demonstrate that the number of partitions of n into at most k parts is equal to the number of partitions of n into parts of size at most k. This proof uses *conjugate* partitions. The conjugate partition λ' of λ is the partition corresponding to the reflection of the Ferrers diagram of λ about the main diagonal. For example, the Ferrers diagram of the conjugate partition of $(6, 3, 3, 1)$ is:

$$\begin{array}{llll}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \\
\bullet & \bullet & \bullet & \\
\bullet & & & \\
\bullet & & & \\
\bullet & & &
\end{array}$$

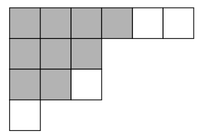
which corresponds to the partition $(4, 3, 3, 1, 1, 1)$. The bijection between λ and λ' implies that the number of partitions of n into at most k parts is equal to the number of partitions of n into parts of size at most k.

Replacing the points of the Ferrers graph of a partition by squares gives the *Young diagram* of the partition.

For example the Young diagram of $(6, 3, 3, 1)$ is:

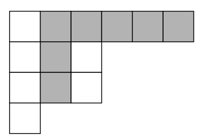


Young diagrams are named after Alfred Young who introduced them in 1900 [12]. If $\mu \subseteq \lambda$ then the diagram of squares that are in the Young diagram of λ and not in the contained Young diagram of μ is called a *skew diagram*, denoted $\lambda - \mu$. The number of squares in the skew diagram $\lambda - \mu$ is denoted by $|\lambda - \mu|$ which is equal to $|\lambda| - |\mu|$. For example, if $\lambda = (6, 3, 3, 1)$ and $\mu = (4, 3, 2)$ then in the diagram below:



the grey squares give the Young diagram of μ, the white squares give the skew diagram $\lambda - \mu$ and the combination of both grey and white squares gives the Young digram of λ. If λ and μ are partitions such that $\mu \prec \lambda$ then $\lambda - \mu$ is called a *horizontal strip* and any column of squares in $\lambda - \mu$ contains at most one square.

Given a Young diagram of a partition λ, if we take a sub-diagram consisting of a square in row i and column j plus all the squares to the right in row i plus all the squares underneath in column j then we have a *hook* in λ which is denoted by $H_{i,j}(\lambda)$. The number of squares in a hook $H_{i,j}(\lambda)$ is denoted by $h_{i,j}(\lambda)$. For example, the grey squares in the Young diagram below is the hook $H_{1,2}(6, 3, 3, 1)$:



The partition $(4, 4, 3, 3, 3, 1)$ has two parts of size 4, three parts of size 3 and one part of size 1. For any partition λ let $m_i(\lambda)$ denote the number of parts of size i. The number $m_i(\lambda)$ is referred to as the *multiplicity* of i in λ. If $\lambda \vdash n$ then n may be expressed as the sum $\sum_{i\geq 1} i \cdot m_i(\lambda)$. Conversely, if $n = \sum_{i=1}^{k} i \cdot m_i$ with non-negative integers $m_1, \ldots, m_k$ then a partition λ of n, in which parts of size i occur m_i times, can be uniquely determined. Therefore, the number of partitions of n is equal to the number of ways of expressing n as the sum $\sum_{i\geq 1} i \cdot m_i$ where m_i are non-negative integers.

Given a non-negative integer n, how many partitions of n are there? Let $p(n)$ denote the number of partitions of a non-negative integer n. For example $p(4) = 5$, i.e., there are five partitions of the number 4 which are (4), $(3, 1)$, $(2, 2)$ $(2, 1, 1)$ and $(1, 1, 1, 1)$. We define $p(0) = 1$ where the partition of 0 is called the *empty partition.* In the 1740s Leonard Euler observed that the generating function for partitions $\sum_{n=0}^{\infty} p(n)q^n$ may be expressed as an infinite product.

Theorem 1.1.1. *The generating function for integer partitions is*

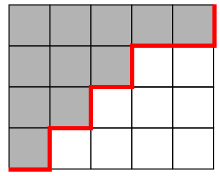
$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1-q^i}. \tag{1.1}$$

Proof. Recall that the number of partitions of n is equal to the number of ways of expressing n as the sum $\sum_{i\geq 1} i \cdot m_i$ where m_i is a non-negative integer for all i. This implies

$$\sum_{n=0}^{\infty} p(n)q^n = \sum_{\lambda} q^{|\lambda|} = \sum_{m_1, m_2, \ldots \geq 0} q^{\sum_{i=0}^{\infty} i \cdot m_i} = \prod_{i=1}^{\infty} \left(\sum_{m_i=0}^{\infty} q^{i \cdot m_i} \right) = \prod_{i=1}^{\infty} \frac{1}{1-q^i}$$

where the last equality follows from the geometric series $\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$. ☐

We will now examine *bounded* partitions, that is partitions with r or less parts and with parts at most c for non-negative integers r and c. Let (c^r) denote the partition $(c, c, \ldots, c)$ where c occurs r times. How many partitions λ are there such that $\lambda \subseteq (c^r)$? Consider the Young diagram of (5^4) with the grey squares representing the Young diagram of $(5, 3, 2, 1)$:

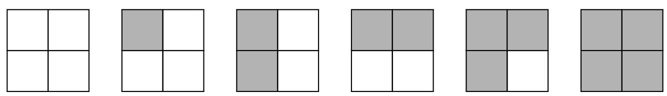


This creates the red path from the bottom left-hand corner to the top right-hand corner of the Young diagram of (5^4) where each step of the path is either to the right or up. We will call such a path a *lattice path*. Any $\lambda \subseteq (5^4)$ will give a lattice path. Also, the squares to the left of any lattice path in the Young diagram of (5^4) will correspond to a partition $\lambda \subseteq (5^4)$. Therefore, the number of partitions $\lambda \subseteq (c^r)$ is the number of lattice paths in the Young diagram of (c^r). Any such lattice path is a sequence of $c+r$ steps, where c of the steps are to the right and r of the steps are up. Therefore, the number of these lattice paths is the number of ways of choosing r upward steps from a total of $c+r$ steps which is equal to $\binom{r+c}{r} = \binom{c+r}{c}$.

We now consider the generating function for partitions with at most r parts with parts at most c, namely

$$\sum_{\lambda \subseteq (c^r)} q^{|\lambda|}.$$

Let $f_{c,r}(q)$ denote this sum. For example the grey Young diagrams below give all the partitions with at most 2 parts with parts at most 2:



This implies $f_{2,2}(q) = 1 + q + 2q^2 + q^3 + q^4$. Setting $q = 1$ in $f_{c,r}(q)$ gives the total number of partitions contained in (c^r), namely $f_{c,r}(1) = \binom{r+c}{r}$.

Lemma 1.1.2. *For non-negative integers c and r the generating function $f_{c,r}(q)$ has the following recursive property:*

$$f_{c,r}(q) = f_{c-1,r}(q) + q^c \cdot f_{c,r-1}(q).$$

Proof. There are two disjoint cases for $\lambda \subseteq (c^r)$:

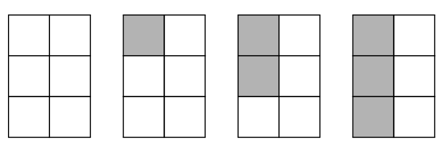
(1) If $\lambda_1 < c$ then $\lambda \subseteq \big((c-1)^r\big)$ and the generating function for such partitions is $f_{c-1,r}(q)$.

(2) If $\lambda_1 = c$ then $(\lambda_2, \lambda_3, \ldots)$ is a partition contained in (c^{r-1}). Therefore, the generating function for partitions $\lambda \subseteq (c^r)$ with $\lambda_1 = c$ is $q^c \cdot f_{c,r-1}(q)$.

Therefore, the generating function for partitions contained in (c^r) is equal to $f_{c-1,r}(q) + q^c \cdot f_{c,r-1}(q)$. □

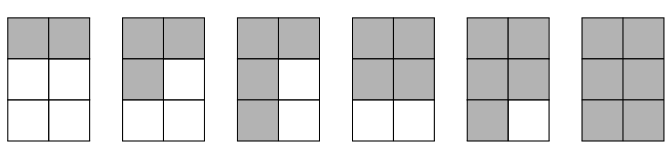
For example we can calculate $f_{2,3}(q)$ by examining the two disjoint cases for $\lambda \subseteq (2^3)$:

(1) If $\lambda_1 < 2$ then the Young diagram of λ can be contained in the first column of the Young diagram of (2^3). Thus all such partitions are given by the following grey Young diagrams:



The generating function for these partitions is $f_{1,3}(q) = 1 + q + q^2 + q^3$.

(2) If $\lambda_1 = 2$ then all such partitions are given by the following grey Young diagrams:



Removing the first row gives all the partitions with at most 2 parts with parts at most 2. Therefore, the generating function for partitions $\lambda \subseteq (2^3)$ with $\lambda_1 = 2$ is $q^2 \cdot f_{2,2}(q) = q^2(1 + q + 2q^2 + q^3 + q^4)$.

This implies

$$f_{2,3}(q) = f_{1,3}(q) + q^2 \cdot f_{2,2}(q) = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$

The recursive relation can be used to calculate $f_{c,r}(q)$ from the boundary values $f_{0,r}(q) = f_{c,0}(q) = 1$. These values arise because the only partition with parts less than or equal to 0 or with 0 parts is the empty partition. The bijection between a partition and its conjugate partition implies $f_{c,r}(q) = f_{r,c}(q)$.

Let $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for any positive integer n and let $(a;q)_0 = 1$. For non-negative integers n and m the *q-binomial coefficient* (or *Gaussian polynomial*) is defined as

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \begin{cases} \dfrac{(q;q)_n}{(q;q)_m (q;q)_{n-m}} & \text{if } m \le n \\ 0 & \text{if } m > n. \end{cases}$$

This gives a product form for the generating function of bounded partitions.

Theorem 1.1.3. *For non-negative integers c and r*

$$\sum_{\lambda \subseteq (c^r)} q^{|\lambda|} = \begin{bmatrix} r+c \\ r \end{bmatrix}_q. \tag{1.2}$$

Proof. The q-binomial coefficient $\begin{bmatrix} r+c \\ r \end{bmatrix}_q$ satisfies the same recursive formula and boundary values as the generating function $f_{c,r}(q)$. The same boundary values are satisfied as

$$f_{0,r}(q) = f_{c,0}(q) = \begin{bmatrix} c \\ 0 \end{bmatrix}_q = 1 = \frac{(q;q)_r}{(q;q)_r} = \begin{bmatrix} r \\ r \end{bmatrix}_q.$$

For $c \ge 1$ and $r \ge 1$ we have the recursive formula

$$\begin{aligned} \begin{bmatrix} r+c \\ r \end{bmatrix}_q &= \frac{(q^{c+1};q)_r}{(q;q)_r} \\ &= \left(\frac{1-q^{c+r}}{1-q^r}\right) \frac{(q^{c+1};q)_{r-1}}{(q;q)_{r-1}} \\ &= \left(\frac{1-q^c}{1-q^r} + q^c\right) \frac{(q^{c+1};q)_{r-1}}{(q;q)_{r-1}} \\ &= \frac{(q^c;q)_r}{(q;q)_r} + q^c \frac{(q^{c+1};q)_{r-1}}{(q;q)_{r-1}} \\ &= \begin{bmatrix} r+c-1 \\ r \end{bmatrix}_q + q^c \begin{bmatrix} r-1+c \\ r-1 \end{bmatrix}_q \end{aligned}$$

which is the same recursive formula as $f_{c,r}(q) = f_{c-1,r}(q) + q^c \cdot f_{c,r-1}(q)$. It follows that

$$f_{c,r}(q) = \begin{bmatrix} r+c \\ r \end{bmatrix}_q$$

for all non-negative integers c and r. □

Taking the limit of (1.2) as c goes to infinity gives the generating function for integer partitions with at most r parts, namely

$$\lim_{c\to\infty} \prod_{i=1}^{r} \frac{1-q^{c+i}}{1-q^i} = \prod_{i=1}^{r} \frac{1}{1-q^i} = \frac{1}{(q;q)_r}. \tag{1.3}$$

Because $f_{c,r}(q) = f_{r,c}(q)$ it follows that (1.3) is also the generating function for integer partitions with parts at most r. Letting r in (1.3) tend to infinity gives the expected generating function (1.1) for integer partitions.

1.2 Plane Partitions

The concept of generalising integer partitions was introduced by Percy A. MacMahon in a sequence of papers titled "Memoir on the theory of the partitions of numbers" which consists of seven parts [13]. In part one, written in 1895, he would consider stacking Ferrers graphs to construct three dimensional structures of dots. MacMahon would refer to one such construction as a three-dimensional graph of a partition, which he would later refer to as a *plane partition*. It is simple to view plane partitions as a kind of (Young) *tableaux* which, like Young diagrams, are named after Alfred Young who introduced them in 1900 [12]. Let λ be an integer partition. A tableau T of shape λ is a filling, using positive integers, of the squares of the Young diagram corresponding to λ. For example:

1	1	1	2	3	3
1	1	3			
5	4	2			
2					

is a tableau of shape $(6, 3, 3, 1)$. If the filling is required to be weakly decreasing both downwards and to the right then this gives a plane partition π of shape λ. For example:

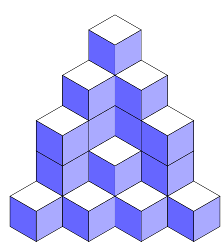
7	6	5	5	4	1
7	5	3			
4	3	3			
2					

gives a plane partition of shape $(6, 3, 3, 1)$. If the filling is strictly decreasing downwards and weakly decreasing to the right then it gives a *column strict* plane partition. This representation of π using a tableau is called a *planar representation* of π. A plane partition is called *symmetric* if its planar representation is unchanged under reflection about the main diagonal. For example:

5	4	3	1
4	2	1	
3	1		
1			

(1.4)

is a symmetric plane partition of shape $(4, 3, 2, 1)$. The numbers in the squares of the planar representation of π are called the *parts* of π and $|\pi|$ denotes the sum of the parts. If n is a non-negative integer such that $n = |\pi|$ then π is a *plane partition of* n. This representation can be thought of as a top-down view of π where each part denotes the number of unit cubes in a stack. This gives a three dimensional representation for plane partitions. For example, the symmetric plane partition (1.4) corresponds to:



If the parts of π are at most t and $\lambda \subseteq (c^r)$ then we say π is *contained in* the *box* $\mathcal{B}(r, c, t)$, which we denote $\pi \subseteq \mathcal{B}(r, c, t)$. Any integer partition has a natural corresponding plane partition which is given by filling its Young diagram with 1s.

As we did with integer partitions, we would like a generating function for bounded and unbounded plane partitions. The generating function for unbounded plane partitions was conjectured by MacMahon in the first part of his paper "Memoir on the theory of the partitions of numbers" and he later proved it in parts five [14] and six [15] which were published in 1912.

Theorem 1.2.1. *The generating function for unbounded plane partitions is*

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{\infty} \frac{1}{(1 - q^i)^i}.$$

In MacMahon's book "Combinatory Analysis" published in 1915 [16] he proved a generating function for bounded plane partitions which is equivalent to the following Theorem.

Theorem 1.2.2. *The generating function for plane partitions contained in $\mathcal{B}(r,c,t)$ is*

$$\sum_{\pi\subseteq\mathcal{B}(r,c,t)} q^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1-q^{i+j+t-1}}{1-q^{i+j-1}}.$$

MacMahon also conjectured a generating function for bounded symmetric plane partitions in his 1898 paper "Partitions of numbers whose graphs possess symmetry" [17], although he was not able to prove this.

Theorem 1.2.3. *The generating function for symmetric plane partitions contained in $\mathcal{B}(r,r,t)$ is*

$$\sum_{\pi\subseteq\mathcal{B}(r,r,t)} q^{|\pi|} = \prod_{i=1}^{r}\frac{1-q^{t+2i-1}}{1-q^{2i-1}}\prod_{1\le i<j\le r}\frac{1-q^{2(t+i+j-1)}}{1-q^{2(i+j-1)}}$$

where the sum is over symmetric plane partitions.

This was proven independently by George E. Andrews in 1978 and Ian G. Macdonald in 1979. Andrews used determinants and *hypergeometric series* for his proof [5] [6]. Andrews build upon earlier work of Basil Gordon [18] and of Edward A. Bender and Donald E. Knuth [19]. Gordon had already shown that one could try to enumerate column strict plane partitions of odd heights. Bender and Knuth computed the generating function of these column strict plane partitions and obtained two (they had to distinguish even and odd bounds) determinants. However, they did not know how to evaluate these determinants. It was Andrews who evaluated these determinants. Macdonald solved the problem in a very different manner by using *symmetric functions* with his proof given in the book "Symmetric Functions and Hall Polynomials" [7]. We shall take the approach of Macdonald and express the claimed generating functions of plane partitions as sums over *Schur functions*.

1.3 Schur Functions

In this section we will use *symmetric* functions to study plane partitions. A function is symmetric if any permutation of its variables leaves it unchanged. Schur functions (also known as Schur polynomials) are a class of symmetric functions that will be used extensively in the study of plane partitions. We will use Schur functions to give a generating function for *semistandard* (Young) tableaux (which are Young tableaux where the filling is strictly increasing downwards and weakly increasing to the right). We will then establish bijections between semistandard tableaux and plane partitions. This will then give generating functions for plane partitions.

Before introducing Schur functions we should be familiar with the *symmetric group* S_n on n letters. It is the group consisting of the set of all permutations on n letters together with the group operation of composition. Let $s_i \in S_n$ denote an *adjacent transposition* which swaps the ith and $(i+1)$th letters, where $1\le i<n$. Also let $s_{ij}\in S_n$ denote a *transposition* swapping the ith and jth letters, where $1\le i<j\le n$. The group S_n is generated by the set of adjacent transpositions [20], i.e., any permutation on n letters can be expressed as the composition of adjacent transpositions. If f is a function in n variables then for $w\in S_n$ we denote $(wf)(x_1,x_2,\dots,x_n) = f(x_{w(1)},x_{w(2)},\dots,x_{w(n)})$. For example, $(s_if)(x_1,x_2,\dots,x_n) = f(x_1,\dots,x_{i-1},x_{i+1},x_i,x_{i+2},\dots,x_n)$. Therefore, f is symmetric if $(s_if)(x_1,x_2,\dots,x_n) = f(x_1,x_2,\dots,x_n)$ for all $i=1,2,\dots,n-1$. Another type of function we will see are *alternating* functions, which are functions that change sign under any transposition of distinct variables. That is, f is an alternating function in n variables if $(s_if)(x_1,x_2,\dots,x_n) = -f(x_1,x_2,\dots,x_n)$ for all $i=1,2,\dots,n-1$.

Schur functions (also known as Schur polynomials) were introduced by Augustin-Louis Cauchy in 1815 [4] and named after Issai Schur. The Schur function s_λ in n variables indexed by a partition λ with $\ell(\lambda)\le n$ is defined as

$$s_\lambda(x_1,x_2,\dots,x_n) = \frac{\det_{1\le i,j\le n}(x_i^{\lambda_j+n-j})}{\det_{1\le i,j\le n}(x_i^{n-j})}. \tag{1.5}$$

If $\ell(\lambda)>n$ then we define $s_\lambda(x_1,x_2,\dots,x_n)=0$. Note that in zero variables we have $s_\lambda(-)=\delta_{\lambda,0}$ where 0 is the empty partition and $\delta_{\lambda,0}$ is the Kronecker delta

$$\delta_{\lambda,\mu} := \begin{cases} 1, & \text{if } \lambda=\mu \\ 0, & \text{otherwise.}\end{cases}$$

The determinant in the denominator of (1.5) is known as the Vandermonde determinant and is equal to the product $\prod_{1\leq i<j\leq n}(x_i - x_j)$, [21]. Cauchy proved that the Schur function is a symmetric polynomial [4]. We will see this by the following lemma.

Lemma 1.3.1. *If $f(x_1, x_2, \ldots, x_n)$ is an alternating polynomial of degree d then*

$$\frac{f(x_1, x_2, \ldots, x_n)}{\prod_{1\leq i<j\leq n}(x_i - x_j)} \tag{1.6}$$

is a symmetric polynomial of degree $d - \frac{n(n-1)}{2}$.

Proof. We will first show that (1.6) is a polynomial. Because f is an alternating polynomial, interchanging the variables x_i and x_j, where $1 \leq i < j \leq n$, changes the sign of $f(x_1, x_2, \ldots, x_n)$, namely

$$(s_{ij}f)(x_1, x_2, \ldots, x_n) = -f(x_1, x_2, \ldots, x_n).$$

If we let $x_i = x_j$ then it follows that $f(x_1, x_2, \ldots, x_n) = 0$. This implies that $x_i = x_j$ is a root of $f(x_1, x_2, \ldots, x_n)$ for any $1 \leq i < j \leq n$. Therefore,

$$f(x_1, x_2, \ldots, x_n) = g(x_1, x_2, \ldots, x_n) \prod_{1\leq i<j\leq n}(x_i - x_j)$$

for some polynomial g of degree $d - \frac{n(n-1)}{2}$. Therefore, (1.6) is a polynomial of degree $d - \frac{n(n-1)}{2}$. Moreover, since both $f(x_1, x_2, \ldots, x_n)$ and $\prod_{1\leq i<j\leq n}(x_i - x_j)$ are alternating polynomials it follows that (1.6) is a symmetric polynomial. □

Corollary 1.3.2. *If $\ell(\lambda) \leq n$ then the Schur function s_λ in n variables is a homogeneous symmetric polynomial of degree $|\lambda|$.*

Proof. By the Vandermonde determinant

$$s_\lambda(x_1, x_2, \ldots, x_n) = \frac{\det_{1\leq i,j\leq n}(x_i^{\lambda_j+n-j})}{\prod_{1\leq i<j\leq n}(x_i - x_j)}. \tag{1.7}$$

The determinant $\det_{1\leq i,j\leq n}(x_i^{\lambda_j+n-j})$ is an alternating polynomial since swapping any two distinct columns causes it to change sign and it is equal to a homogeneous polynomial of degree

$$(n-1+\lambda_1) + (n-2+\lambda_2) + \cdots + (\lambda_n) = |\lambda| + \frac{n(n-1)}{2}.$$

Since the Vandermonde determinant is homogeneous, it follows that (1.7) is a homogeneous symmetric polynomial of degree $|\lambda|$. □

We will now see how the Schur function gives rise to a generating function for semistandard tableaux. We say that a semistandard tableau T has *content* $\mu = (\mu_1, \mu_2, \ldots)$ if μ_j is the number of occurrences of the number j in the filling of T for all j. For example:

$$\begin{array}{|c|c|c|c|c|c|}\hline 1&1&1&2&3&3\\\hline 2&3&3\\\cline{1-3} 5&5&6\\\cline{1-3} 7\\\cline{1-1}\end{array} \tag{1.8}$$

is a semistandard tableau of content $(3, 2, 4, 0, 2, 1, 1)$. Here, as in the case of partitions, the infinite string of zeros is usually omitted. If a semistandard tableau T has content μ then it has an associated monomial $x^\mu = x_1^{\mu_1}x_2^{\mu_2}\cdots$, which, by abuse of notation, is often represented as x^T. Hence (1.8) has associated monomial $x_1^3x_2^2x_3^4x_5^2x_6x_7$. We denote the set of all semistandard tableaux of shape λ and content μ by $\mathrm{SSYT}(\lambda, \mu)$. If the content is unrestricted then $\mathrm{SSYT}(\lambda, \cdot)$ denotes the set of all semistandard tableaux of shape λ. The set $\mathrm{SSYT}_n(\lambda, \cdot)$ denotes all semistandard tableaux of shape λ whose filling uses integers at most n.

Example 1.3.3. *The semistandard tableaux in $\mathrm{SSYT}_3((2, 1, 1), \cdot)$, with their corresponding monomials underneath, are:*

$$\begin{array}{ccc} \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \end{array} \\ x_1^2x_2x_3 & x_1x_2^2x_3 & x_1x_2x_3^2. \end{array}$$

Hence, we have:

$$\sum_{T\in\mathrm{SSYT}_3((2,1,1),\cdot)} x^T = x_1^2x_2x_3 + x_1x_2^2x_3 + x_1x_2x_3^2.$$

It happens that the polynomial $x_1^2x_2x_3+x_1x_2^2x_3+x_1x_2x_3^2$ from the previous example is equal to $s_{(2,1,1)}(x_1,x_2,x_3)$. This is no coincidence.

Lemma 1.3.4 (Branching Rule)**.** *Let λ be a partition. Then for $n \geq 1$ we have*

$$s_\lambda(x_1,x_2,\ldots,x_n) = \sum_{\mu\prec\lambda} s_\mu(x_1,x_2,\ldots,x_{n-1})x_n^{|\lambda-\mu|}. \tag{1.9}$$

Proof. If $\ell(\lambda) > n$ then $\ell(\mu) > n-1$ for all $\mu \prec \lambda$, which implies that (1.9) holds when $\ell(\lambda) > n$ as both sides will be 0. Now suppose that $\ell(\lambda) \leq n$. A revision on the properties of the determinant can be found in [21]. We begin by subtracting the last row of the determinant in the numerator of (1.5) from all of the other rows:

$$\begin{aligned} s_\lambda(x_1,x_2,\ldots,x_n) &= \frac{\det_{1\leq i,j\leq n}(x_i^{\lambda_j+n-j})}{\prod_{1\leq i<j\leq n}(x_i-x_j)} \\ &= \frac{\det_{1\leq i,j\leq n}\left(\begin{cases} x_i^{\lambda_j+n-j} - x_n^{\lambda_j+n-j} & \text{if } i<n \\ x_n^{\lambda_j+n-j} & \text{if } i=n \end{cases}\right)}{\prod_{1\leq i<j<n}(x_i-x_j)\prod_{i=1}^{n-1}(x_i-x_n)}. \end{aligned}$$

We now distribute the product $\prod_{i=1}^{n-1}(x_i-x_n)$ into the determinant in the numerator by dividing row i by (x_i-x_n) for all $i=1,2,\ldots,n-1$:

$$\begin{aligned} s_\lambda(x_1,x_2,\ldots,x_n) &= \frac{\det_{1\leq i,j\leq n}\left(\begin{cases} (x_i^{\lambda_j+n-j} - x_n^{\lambda_j+n-j})/(x_i-x_n) & \text{if } i<n \\ x_n^{\lambda_j+n-j} & \text{if } i=n \end{cases}\right)}{\prod_{1\leq i<j<n}(x_i-x_j)} \\ &= \frac{\det_{1\leq i,j\leq n}\left(\begin{cases} x_n^{\lambda_j-j}\sum_{k=0}^{\lambda_j+n-j-1} x_i^k x_n^{n-k-1} & \text{if } i<n \\ x_n^{\lambda_j+n-j} & \text{if } i=n \end{cases}\right)}{\prod_{1\leq i<j<n}(x_i-x_j)} \\ &= \frac{\det_{1\leq i,j\leq n}\left(\begin{cases} \sum_{k=0}^{\lambda_j+n-j-1} x_i^k x_n^{n-k-1} & \text{if } i<n \\ 1 & \text{if } i=n \end{cases}\right)\prod_{j=1}^n x_n^{\lambda_j-j+1}}{\prod_{1\leq i<j<n}(x_i-x_j)}. \end{aligned} \tag{1.10}$$

We will now proceed by successively subtracting columns in the determinant in the numerator: the first minus the second, then the second minus the third, etc., up to the $(n-1)$th minus the nth. This will result in the bottom row having 0 in columns one to $n-1$ and 1 in column n. This implies that (1.10) is equal to

$$\frac{\det_{1\leq i,j\leq n-1}\left(\sum_{k=\lambda_{j+1}}^{\lambda_j} x_i^{k+n-j-1}x_n^{j-k}\right)\prod_{j=1}^n x_n^{\lambda_j-j+1}}{\prod_{1\leq i<j<n}(x_i-x_j)}. \tag{1.11}$$

It follows from the multilinearity of the determinant that (1.11) is equal to

$$\begin{aligned}\sum_{\mu\prec\lambda}\frac{\det_{1\le i,j\le n-1}\left(x_i^{\mu_j+n-j-1}x_n^{j-\mu_j}\right)\prod_{j=1}^n x_n^{\lambda_j-j+1}}{\prod_{1\le i<j\le n-1}(x_i-x_j)} &= \sum_{\mu\prec\lambda}\frac{\det_{1\le i,j\le n-1}\left(x_i^{\mu_j+n-j-1}\right)x_n^{\lambda_n-n+1}\prod_{j=1}^{n-1}x_n^{\lambda_j-\mu_j+1}}{\prod_{1\le i<j\le n-1}(x_i-x_j)}\\ &= \sum_{\mu\prec\lambda}\frac{\det_{1\le i,j\le n-1}\left(x_i^{\mu_j+n-j-1}\right)x_n^{\sum_{j=1}^n\lambda_j-\mu_j}}{\prod_{1\le i<j\le n-1}(x_i-x_j)}\\ &= \sum_{\mu\prec\lambda}\frac{\det_{1\le i,j\le n-1}\left(x_i^{\mu_j+n-j-1}\right)x_n^{|\lambda-\mu|}}{\prod_{1\le i<j\le n-1}(x_i-x_j)}\\ &= \sum_{\mu\prec\lambda}s_\mu(x_1,x_2,\ldots,x_{n-1})x_n^{|\lambda-\mu|}. \qquad\square\end{aligned}$$

Theorem 1.3.5. *Let λ be a partition. Then*

$$s_\lambda(x_1,x_2,\ldots,x_n)=\sum_{T\in\mathrm{SSYT}_n(\lambda,\cdot)}x^T. \tag{1.12}$$

Proof. If $\ell(\lambda)>n$ then the Young diagram of λ will contain at least $n+1$ rows and there is no way to fill the Young diagram using integers at most n such that the filling is strictly decreasing downwards. Thus, if $\ell(\lambda)>n$ then (1.12) will hold as both sides are 0. We will now suppose that $\ell(\lambda)\le n$ and proceed by induction on n. If $n=0$ then λ can only be the empty partition and the only tableau of shape λ has the associated monomial 1. Hence, (1.12) holds when $n=0$. Assume that (1.12) holds for all n up to $k-1$ for some $k\ge 2$. Let λ be a partition of length at most k. If T is a semistandard tableau of shape $\mu\prec\lambda$ whose filling uses integers from $\{1,2,\ldots,k-1\}$ then filling each square of $\lambda-\mu$ with a k and adjoining it to T gives a semistandard tableau of shape λ whose filling uses integers from $\{1,2,\ldots,k\}$ with corresponding monomial $x^Tx_k^{|\lambda-\mu|}$. Conversely removing any squares filled by a k in a semistandard tableau of shape λ with squares filled from $\{1,2,\ldots,k\}$ gives a a semistandard tableau of shape μ for some $\mu\prec\lambda$ whose filling uses integers from $\{1,2,\ldots,k-1\}$. Therefore,

$$\sum_{T\in\mathrm{SSYT}_k(\lambda,\cdot)}x^T=\sum_{\mu\prec\lambda}\left(x_k^{|\lambda-\mu|}\sum_{T\in\mathrm{SSYT}_{k-1}(\mu,\cdot)}x^T\right).$$

The inductive hypothesis and branching rule gives the required result:

$$\sum_{T\in\mathrm{SSYT}_k(\lambda,\cdot)}x^T=\sum_{\mu\prec\lambda}\left(x_k^{|\lambda-\mu|}\sum_{T\in\mathrm{SSYT}_{k-1}(\mu,\cdot)}x^T\right)=\sum_{\mu\prec\lambda}s_\mu(x_1,x_2,\ldots,x_{k-1})x_k^{|\lambda-\mu|}=s_\lambda(x_1,x_2,\ldots,x_k). \qquad\square$$

We will now use our generating function for semistandard tableaux to construct a generating function for plane partitions. Tableaux and plane partitions are both fillings of Young diagrams. It is easy to translate a semistandard tableau to a column strict plane partition and vice versa. Given a semistandard tableau T we can replace each number t in the filling of T by x_t to obtain an alternative representation of T. For example, the tableau (1.8) can be represented as:

$$\begin{array}{|c|c|c|c|c|c|}\hline x_1&x_1&x_1&x_2&x_3&x_3\\\hline x_2&x_3&x_3\\\cline{1-3} x_5&x_5&x_6\\\cline{1-3} x_7\\\cline{1-1}\end{array} \tag{1.13}$$

We can let each variable represent a stack of cubes to give a column strict plane partition. For example, if we let $x_i=q^{8-i}$ in (1.13) then by reading the powers of q gives a column strict plane partition, namely:

$$\begin{array}{|c|c|c|c|c|c|}\hline x_1&x_1&x_1&x_2&x_3&x_3\\\hline x_2&x_3&x_3\\\cline{1-3} x_5&x_5&x_6\\\cline{1-3} x_7\\\cline{1-1}\end{array}\ \mapsto\ \begin{array}{|c|c|c|c|c|c|}\hline q^7&q^7&q^7&q^6&q^5&q^5\\\hline q^6&q^5&q^5\\\cline{1-3} q^3&q^3&q^2\\\cline{1-3} q\\\cline{1-1}\end{array}\ \mapsto\ \begin{array}{|c|c|c|c|c|c|}\hline 7&7&7&6&5&5\\\hline 6&5&5\\\cline{1-3} 3&3&2\\\cline{1-3} 1\\\cline{1-1}\end{array}$$

The power of q in x^T is equal to the sum of the parts of the corresponding column strict plane partition. This implies that the generating function for column strict plane partitions of shape (c^r) with parts at most t is

$$s_{(c^r)}(q^t, q^{t-1}, \ldots, q).$$

Because the Schur function is homogeneous it follows that $s_\lambda(zx_1, zx_2, \ldots, zx_n) = z^{|\lambda|} s_\lambda(x_1, x_2, \ldots x_n)$. This implies $s_{(c^r)}(q^t, q^{t-1}, \ldots, q) = q^{cr} s_{(c^r)}(q^{t-1}, q^{t-2}, \ldots, 1)$. This allows us to form a generating function for bounded plane partitions.

Theorem 1.3.6. *The generating function for plane partitions contained in $\mathcal{B}(r, c, t)$ is:*

$$\sum_{\pi \subseteq \mathcal{B}(r,c,t)} q^{|\pi|} = q^{-c\binom{r}{2}} s_{(c^r)}(q^{t+r-1}, q^{t+r-2}, \ldots, 1).$$

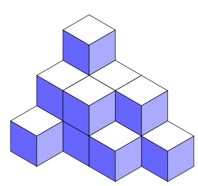
Proof. We will establish a bijection between column strict plane partitions of shape (c^r) with parts at most $t + r$ and plane partitions contained in $\mathcal{B}(r, c, t)$. Let $\lambda = (c^r)$ and let π be a column strict plane partition of shape λ with parts at most $t + r$. If we reduce each part in row $r - k + 1$ by k, in the planar representation of π, for each $k \in \{1, 2, \ldots, r\}$ then we obtain a plane partition contained in $\mathcal{B}(r, c, t)$. For example, if $c = 6$, $r = 4$ and $t = 3$ then:

7	7	7	6	5	5
6	5	5	5	4	3
4	3	2	2	2	2
2	2	1	1	1	1

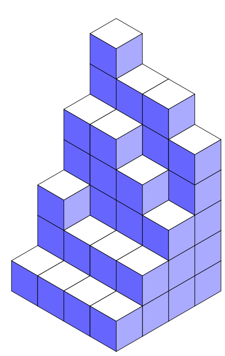
is a column strict plane partition of shape (c^r) with parts at most $t + r$ and removing 4 from each part in the first row, 3 from each part in the second row, etc., gives the plane partition:

3	3	3	2	1	1
3	2	2			
2	1				
1	1				

which is contained in $\mathcal{B}(r, c, t)$. Conversely, given a plane partition $\pi' \subseteq \mathcal{B}(r, c, t)$ if we add k cubes to each stack (including empty stacks) in row $r - k + 1$, in the three dimensional representation of π', for each $k \in \{1, 2, \ldots, r\}$ then we obtain a column strict plane partition of shape λ with parts at most $t + r$. For example, if $r = c = t = 4$ then:



is a plane partition contained in $\mathcal{B}(r, c, t)$. Adding 1 cube to each stack in row 4, 2 cubes to each stack in row 3, etc., we obtain the column strict plane partition of shape (c^r) with parts at most $t + r$:



Therefore, by this bijection, each plane partition in $\mathcal{B}(r, c, t)$ can be uniquely obtained from some column strict plane partition of shape (c^r) with parts at most $t + r$. Such a plane partition will have $cr(r + 1)/2 = c\binom{r+1}{2}$ less cubes than its corresponding column strict plane partition. Since the generating function for column strict plane partitions of shape (c^r) with parts at most $t+r$ is $q^{cr}s_{(c^r)}(q^{t+r-1}, q^{t+r-2}, \ldots, 1)$ it follows that the generating function for plane partitions contained in $\mathcal{B}(r,c,t)$ is

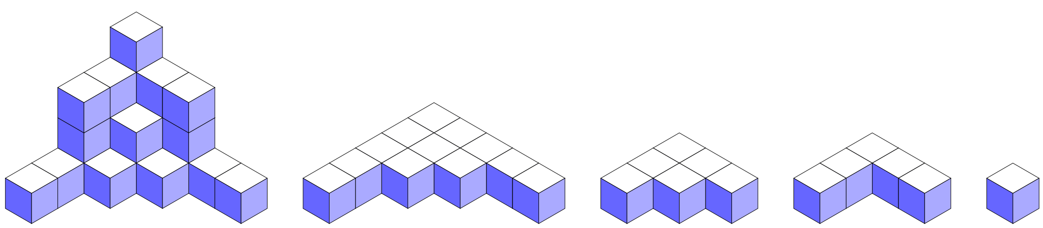
$$q^{cr-c\binom{r+1}{2}}s_{(c^r)}(q^{t+r-1}, q^{t+r-2}, \ldots, 1) = q^{-c\binom{r}{2}}s_{(c^r)}(q^{t+r-1}, q^{t+r-2}, \ldots, 1). \qquad \square$$

A similar bijection also exists between symmetric plane partitions contained in $\mathcal{B}(r,r,t)$ and column strict plane partitions with odd parts contained in $\mathcal{B}(r,t,2r-1)$. We will establish this bijection in two different ways, with the first being the classical construction. Let $\pi \subseteq \mathcal{B}(r,r,t)$ be a symmetric plane partition and let π_k denote the kth layer of cubes in the three dimensional representation of π. Then π_1 consists of the cubes forming the first (base) layer of π, π_2 consists of the cubes forming the second layer (the layer above the first), etc. For example, if we let π be the symmetric plane partition given by:

4	3	3	1	1
3	2	1		
3	1			
1				
1				

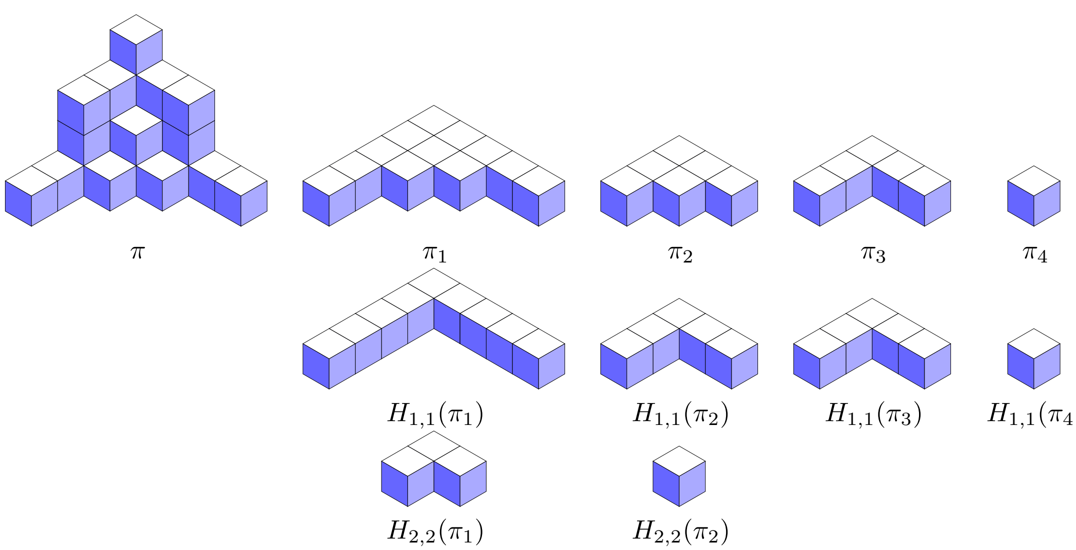
(1.14)

then we have:

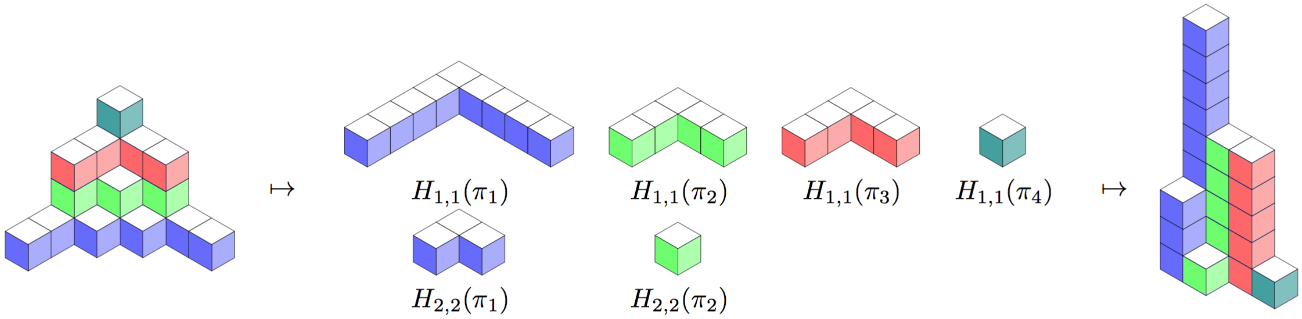


π π_1 π_2 π_3 π_4

Each layer π_k can be decomposed into hooks about the main diagonal, namely $H_{1,1}(\pi_k)$, $H_{2,2}(\pi_k)$, etc. For example:

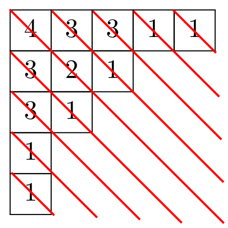


As π is a symmetric plane partition it follows that each layer is also a symmetric plane partition which implies each hook contains an odd number of cubes. Moreover, $2r-1 \geq h_{1,1}(\pi_k) > h_{2,2}(\pi_k) > \cdots$ for all k. Mapping each hook $H_{i,i}(\pi_j)$ to a stack of $h_{i,i}(\pi_j)$ cubes with base at position $(i,j,1)$ gives a column strict plane partition with stacks of odd height contained in $\mathcal{B}(r,t,2r-1)$. For example:



This process is easily reversible and establishes the required bijection.

A modern approach to this bijection is to use diagonal slices. For example we may diagonally slice the symmetric plane partition (1.14) as follows:

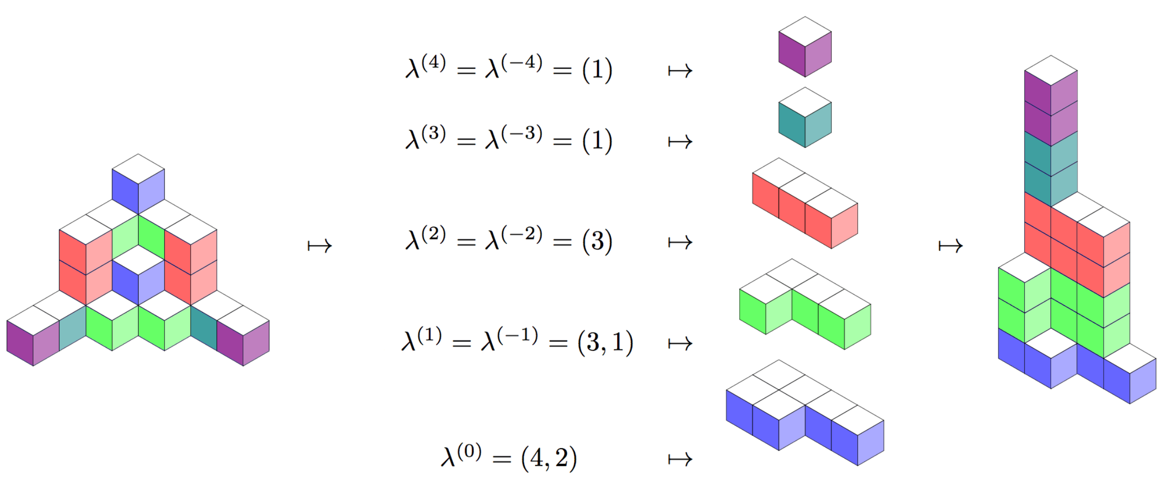


$$(1.15)$$

Let $\pi \subseteq \mathcal{B}(r,r,t)$ be a symmetric plane partition. Reading along each slice of π gives an integer partition, where $\lambda^{(0)}$ denotes the integer partition corresponding to the main diagonal slice. For example, the slice along the main diagonal of (1.15) gives the integer partition $(4,2)$. Reading from the bottom left-hand slice to the top right-hand slice of π gives a sequence of interlacing partitions $\cdots \prec \lambda^{(-2)} \prec \lambda^{(-1)} \prec \lambda^{(0)} \succ \lambda^{(1)} \succ \lambda^{(2)} \succ \cdots$. For example, the diagonal slices of (1.15) give the following sequence of interlacing partitions:

$$(1) \prec (1) \prec (3) \prec (3,1) \prec (4,2) \succ (3,1) \succ (3) \succ (1) \succ (1).$$

The sequence will be palindromic since π is symmetric. Each $\lambda^{(i)}$ corresponds to a plane partition, namely the plane partition formed by filling the Young diagram of $\lambda^{(i)}$ with 1s. Stacking the three dimensional representations of the plane partitions corresponding to the $\lambda^{(i)}$s, in the order of the partition with least cubes on the top to partition with most cubes on the bottom, gives a column strict plane partition with odd parts contained in $\mathcal{B}(r,t,2r-1)$. The following is the process completed on (1.14), which gives the same result as the first bijection:



This process is easily reversible and again establishes the required bijection.

Theorem 1.3.7. *The generating function for symmetric plane partitions contained in $\mathcal{B}(r,r,t)$ is*

$$\sum_{\lambda \subseteq (t^r)} s_\lambda(q^{2r-1}, q^{2r-3}, \ldots, q). \tag{1.16}$$

Equivalently this is the generating function for column strict plane partitions with odd parts contained in $\mathcal{B}(r, t, 2r-1)$.

Proof. Let $\lambda \subseteq (t^r)$. Then $s_\lambda(q^{2r-1}, q^{2r-3}, \ldots, q)$ is the generating function for column strict plane partitions of shape λ with odd parts at most $2r-1$. Therefore, summing over all $\lambda \subseteq (t^r)$ gives the generating function (1.16) for column strict plane partitions with odd parts contained in $\mathcal{B}(r, t, 2r-1)$. The bijection between column strict plane partitions with odd parts contained in $\mathcal{B}(r, t, 2r-1)$ and symmetric plane partitions contained in $\mathcal{B}(r, r, t)$ implies that (1.16) is also the generating function for symmetric plane partitions contained in $\mathcal{B}(r, r, t)$. $\square$

1.4 Skew Schur Functions

In this section we will examine the skew Schur functions, which are a generalisation of the ordinary Schur functions. Before giving a definition, we must first generalise the notion of shape of Young tableaux. If λ and μ are integer partitions such that $\mu \subseteq \lambda$ then a tableau T of (skew) shape λ/μ is a filling, using positive integers, of the squares of the skew diagram $\lambda - \mu$. For example

$$\begin{array}{cccccc} & & & & 2 & 4 \\ & & & 8 & 7 & \\ & & & 2 & 5 & \\ 4 & 4 & & & & \\ 6 & & & & & \end{array}$$

is a tableau of shape $(6, 5, 5, 2, 1)/(4, 3, 3)$. If the filling is required to be strictly increasing downwards and weakly increasing to the right then this gives a semistandard tableau of shape λ/μ. For example

$$\begin{array}{ccccc} & & & & 2 \\ & & 4 & 5 & \\ & & 5 & & \\ 1 & 1 & 6 & & \end{array} \tag{1.17}$$

is a semistandard tableau of shape $(5, 4, 3, 3)/(6, 2, 2)$. All definitions related to tableaux of shape λ generalise in the obvious way to tableaux of shape λ/μ. For example, the semistandard tableau (1.17) has content $\eta = (2, 1, 0, 1, 2, 1)$, has associated monomial $x^\eta = x_1^2 x_2 x_4 x_5^2 x_6$ and belongs to the set $\mathrm{SSYT}((5, 4, 3, 3)/(6, 2, 2), \eta)$.

We can also generalise the Schur function to allow indexing by skew shapes. For finite alphabets x and y in m and n variables, respectively, we define the Schur function indexed by skew shape λ/μ to be the function such that

$$s_\lambda(x, y) = \sum_{\mu \subseteq \lambda} s_{\lambda/\mu}(y) s_\mu(x) \tag{1.18}$$

if $n \geq \ell(\lambda) - \ell(\mu)$. Otherwise, $s_{\lambda/\mu}(y) := 0$ if $n < \ell(\lambda) - \ell(\mu)$. We will now use the Branching Rule, see Lemma 1.3.4, to give $s_{\lambda/\mu}(y)$ explicitly in the case $n \geq \ell(\lambda) - \ell(\mu)$. Let y be the alphabet in the n variables $y_1, y_2, \ldots, y_n$. Clearly if $n = 0$ then by definition (1.18) we have $s_\lambda(-) = \delta_{\lambda,\mu}$. So now suppose that $n \geq 1$. It follows immediately by the Branching Rule and induction on n that

$$s_\lambda(x, y) = \sum_{\substack{\eta^{(0)} \prec \eta^{(1)} \prec \cdots \prec \eta^{(n)} \\ \eta^{(0)} = \mu \\ \eta^{(n)} = \lambda}} \left(s_\mu(x) \prod_{i=1}^{n} y_i^{|\eta^{(i)} - \eta^{(i-1)}|} \right) \tag{1.19}$$

where $\eta^{(0)}, \ldots, \eta^{(n)}$ are partitions. Therefore, we obtain an explicit equation for the skew Schur function, in the case $n \geq \ell(\lambda) - \ell(\mu)$,

$$s_{\lambda/\mu}(y) = \sum_{\substack{\eta^{(0)} \prec \eta^{(1)} \prec \cdots \prec \eta^{(n)} \\ \eta^{(0)} = \mu \\ \eta^{(n)} = \lambda}} \left(\prod_{i=1}^{n} y_i^{|\eta^{(i)} - \eta^{(i-1)}|} \right). \tag{1.20}$$

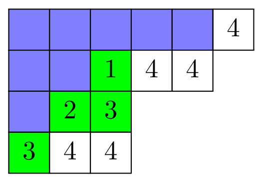
Note that the sum in (1.20) is 0 when $n < \ell(\lambda) - \ell(\mu)$. Hence, (1.20) holds for both cases of n. We have seen that the ordinary Schur functions indexed by partitions are symmetric functions. Therefore, it follows from the definition (1.18) that the skew Schur functions are symmetric functions. In the next Lemma we will see that the skew Schur function satisfies a combinatorial property similar to that of Theorem 1.3.5.

Lemma 1.4.1. *Let λ and μ be partitions such that $\mu \subseteq \lambda$. Then*

$$s_{\lambda/\mu}(x_1, x_2, \ldots, x_n) = \sum_{T \in \mathrm{SSYT}_n(\lambda/\mu, \cdot)} x^T. \tag{1.21}$$

Proof. If $n < \ell(\lambda) - \ell(\mu)$ then the skew diagram $\lambda - \mu$ will have a column with at least $n+1$ rows and we cannot fill these squares using integers at most n such that the filling is strictly decreasing downwards. Thus, if $n < \ell(\lambda) - \ell(\mu)$ then (1.21) holds as both sides will be 0. So we will assume that $n \geq \ell(\lambda) - \ell(\mu)$ and proceed by induction on n. If $n = 0$ then the only defined tableau whose filling uses no integers is the tableau of shape λ/μ with $\lambda = \mu$ which has associated monomial 1. Thus, (1.21) holds for $n = 0$. Assume that (1.21) holds for all n up to $k-1$ for some $k \geq 2$. Let λ be a partition of length at most k. Suppose T is a semistandard tableau of shape ν/μ, where ν is a partition such that $\mu \subseteq \nu \prec \lambda$, whose filling uses integers from $\{1, 2, \ldots, k-1\}$. Then filling each square of $\lambda - \nu$ with a k and adjoining it to T gives a semistandard tableau of shape λ/μ whose filling uses integers from $\{1, 2, \ldots, k\}$ with corresponding monomial $x^T x_k^{|\lambda-\nu|}$. Conversely removing any squares filled by a k in a semistandard tableau of shape λ/μ with squares filled from $\{1, 2, \ldots, k\}$ gives a a semistandard tableau of shape ν/μ, where ν is a partition such that $\mu \subseteq \nu \prec \lambda$, whose filling uses integers from $\{1, 2, \ldots, k-1\}$.

For example, in the diagram:



the filled squares give a semistandard tableau of shape $(6, 5, 3, 3)/(5, 2, 1)$ whose filling uses positive integers at most 4. Removing from this semistandard tableau the squares filled with a 4 gives the semistandard tableau represented by the green squares which has shape $(5, 3, 3, 1)/(5, 2, 1)$ and whose filling uses positive integers at most 3. Moreover, $(5, 2, 1) \subseteq (5, 3, 3, 1) \prec (6, 5, 3, 3)$.

Therefore,

$$\sum_{T \in \mathrm{SSYT}_k(\lambda/\mu, \cdot)} x^T = \sum_{\substack{\nu \prec \lambda \\ \nu \supseteq \mu}} \left(x_k^{|\lambda-\nu|} \sum_{T \in \mathrm{SSYT}_{k-1}(\nu/\mu, \cdot)} x^T \right)$$

and by the inductive hypothesis

$$\begin{aligned}
\sum_{T \in \mathrm{SSYT}_k(\lambda/\mu, \cdot)} x^T &= \sum_{\nu \prec \lambda} \left(\sum_{\substack{\eta^{(0)} \prec \cdots \prec \eta^{(k-1)} \\ \eta^{(0)} = \mu \\ \eta^{(k-1)} = \nu}} \left(\prod_{i=1}^{k-1} x_i^{|\nu^{(i)} - \nu^{(i-1)}|} \right) x_k^{|\lambda-\nu|} \right) \\
&= \sum_{\substack{\eta^{(0)} \prec \cdots \prec \eta^{(k)} \\ \eta^{(0)} = \mu \\ \eta^{(k)} = \lambda}} \left(\prod_{i=1}^{k} x_i^{|\nu^{(i)} - \nu^{(i-1)}|} \right) \\
&= s_{\lambda/\mu}(x_1, x_2, \ldots, x_k).
\end{aligned}$$

In conclusion, (1.21) holds for all n. ☐

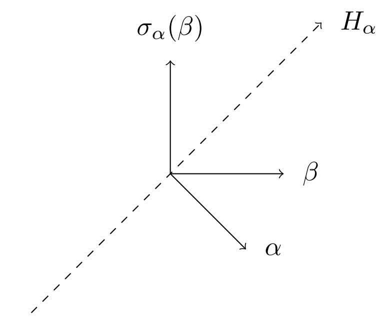
Chapter 2

Root Systems

In Chapter 1 we derived generating functions for plane partitions as sums over Schur functions. In this Chapter we will see how to evaluate some of these sums using representation theory. In particular we will use results from the theory of *root systems*, which has significant applications to the representation theory of semi-simple Lie algebras. Most of the results we use in this Chapter follow immediately from studying root systems on their own and we will not need to reference the underlying representation theory. In Section 2.4 we outline Macdonald's proof of Theorem 1.2.3.

2.1 Root Systems

In this Section we will introduce root systems and give some basic results. For the remainder of this Chapter we let E denote an Euclidean space, i.e., a finite-dimensional vector space over $\mathbb{R}$ with an inner product $(\cdot,\cdot) : E \to \mathbb{R}$. A familiar example of such a space is $\mathbb{R}^n$ together with the standard inner product (also known as the dot product). Given any non-zero vector $\alpha \in E$, the set of orthogonal vectors to α, i.e., $\{\beta \in E : (\alpha, \beta) = 0\}$ determines a *hyperplane* in E and is denoted as H_α. An important notion in the theory of root systems are *reflections*. A reflection $\sigma_\alpha : E \to E$, indexed by a non-zero vector $\alpha \in E$, is an invertible linear transformation that reflects all $\beta \in E$ about the hyperplane H_α. We have the following example in $\mathbb{R}^2$:



The map σ_α has a simple explicit formula, which we give in the following Lemma.

Lemma 2.1.1. *For any non-zero vector $\alpha \in E$ the map $\sigma_\alpha : E \to E$ can be explicitly defined as:*

$$\sigma_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha. \tag{2.1}$$

Proof. Clearly the right-hand side of (2.1) is linear in β. By definition the map σ_α fixes any vector in H_α and maps α to its negative. It will suffice to show that the right-hand side of (2.1) satisfies these two properties, as each vector in E can be written as a linear combination of a vector in H_α and α. For the first property, suppose that β is in the hyperplane H_α. Then as $(\beta, \alpha) = 0$ we have

$$\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha = \beta = \sigma_\alpha(\beta).$$

Therefore, (2.1) holds when $\beta \in H_\alpha$. Moreover, evaluating the right-hand side of (2.1) when $\beta = \alpha$ gives

$$\alpha - \frac{2(\alpha,\alpha)}{(\alpha,\alpha)}\alpha = \alpha - 2\alpha = -\alpha = \sigma_\alpha(\alpha).$$

Hence, (2.1) holds for $\beta = \alpha$. In conclusion (2.1) holds for all $\beta \in E$. Thus, (2.1) also defines its own inverse. $\square$

Throughout this Chapter we will often reference the number $\frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ so we will denote it by $\langle \beta, \alpha \rangle$. We can now define root systems. We say that a subset $\Phi \subseteq E$ is a root system if the following conditions are satisfied:

(R1) The set Φ is finite, does not contain the zero vector and spans E.

(R2) If $\alpha \in \Phi$ then: $c\alpha \in \Phi$ if and only if $c = \pm 1$.

(R3) If $\alpha \in \Phi$ then σ_α leaves Φ invariant.

(R4) If $\alpha, \beta \in \Phi$ then $\langle \beta, \alpha \rangle \in \mathbb{Z}$.

The vectors in Φ are called *roots*. We call $\dim(E)$ the *rank* of Φ. For any $\alpha \in \Phi$ the map σ_α is a permutation of Φ, as Φ is finite and the reflection σ_α leaves Φ invariant. Therefore, the group generated by reflections σ_α forms a subgroup of the symmetric group on Φ. We call this group the *Weyl group* of Φ, which is denoted by $\mathscr{W}$. For the remainder of this Section Φ will denote a root system with rank ℓ and with Weyl group $\mathscr{W}$.

A subset $\Delta \subseteq \Phi$ is called a base if:

(B1) The set Δ is a basis of E.

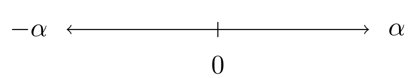
(B2) For each $\beta \in \Phi$ there exist integers k_α either all non-positive or all non-negative such that $\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha$.

Clearly there must exist a subset $\Delta \subseteq \Phi$ satisfying the first condition as Φ spans E. However, the existence of such a subset satisfying the second condition is not-obvious. In fact every root system Φ has a base, a proof can be found in [22]. The elements of Δ are called simple roots. The values of k_α in (B2) are unique. Since if there were numbers c_α such that $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$ then

$$\beta = \sum_{\alpha \in \Delta} k_\alpha \alpha = \sum_{\alpha \in \Delta} c_\alpha \alpha \quad \Rightarrow \quad \sum_{\alpha \in \Delta} (k_\alpha - c_\alpha)\alpha = 0.$$

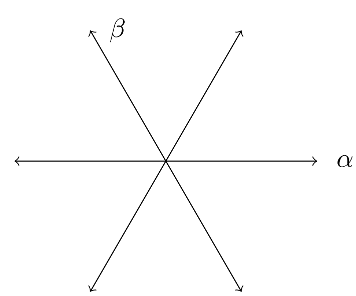
It follows from Δ being a basis of E that Δ is linearly independent. Therefore, $k_\alpha - c_\alpha = 0$ and $k_\alpha = c_\alpha$ for all $\alpha \in \Delta$. This uniqueness means that the height of a root $\beta \in \Phi$, relative to Δ, given by $\mathrm{ht}(\beta) := \sum_{\alpha \in \Delta} k_\alpha$ is well defined. As $0 \notin \Phi \supseteq \Delta$ not all k_α in (B2) can be zero, i.e., $\mathrm{ht}(\beta) \neq 0$. A root β is called positive if $\mathrm{ht}(\beta) > 0$ and negative if $\mathrm{ht}(\beta) < 0$ which we denote by $\beta \succ 0$ and $\beta \prec 0$, respectively. We define the subsets $\Phi^+ \subseteq \Phi$ and $\Phi^- \subseteq \Phi$ to be the disjoint sets of positive and negative roots, respectively. Note that $\Phi = \Phi^+ \cup \Phi^-$. The Weyl group $\mathscr{W}$ is generated by the reflections σ_α with $\alpha \in \Delta$, see [22] for a proof. When we write a reflection σ as the product $\sigma_{\alpha_1} \cdots \sigma_{\alpha_k}$ with each $\alpha_i \in \Delta$ and with k minimal, then k is called the *length* of σ, relative to Δ, which we denote by $\ell(\sigma)$. We will now see two examples of root systems.

Example 2.1.2. *let Φ be a root system in $\mathbb{R}$. If α is a root in Φ then the only other root in Φ is $-\alpha$, this follows from (R2) and the fact that $\mathbb{R}$ is 1-dimensional. Therefore, all such root systems $\Phi \subseteq \mathbb{R}$ are of the form:*



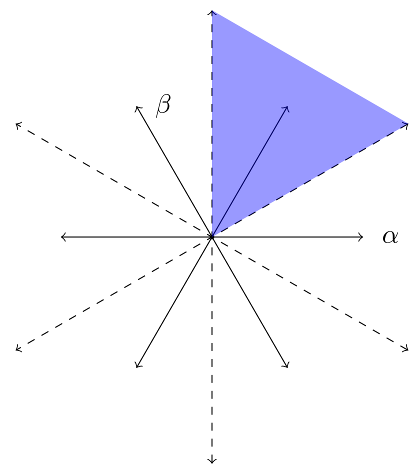
The set $\{\alpha\}$ forms a base for Φ.

Example 2.1.3. *The following diagram corresponds to a root system $\Phi \subseteq \mathbb{R}^2$ with base $\{\alpha, \beta\}$:*



Given a vector $\gamma \in E$ we may like to know if it lies on a hyperplane H_α for some $\alpha \in \Phi$. A vector $\gamma \in E$ is called regular if it is not in any hyperplane, that is $\gamma \in E \backslash \left(\bigcup_{\alpha\in\Phi} H_\alpha\right)$, and is called singular otherwise. The hyperplanes partition the space E into disjoint open subsets of E which are called *Weyl chambers*, i.e., the Weyl chambers are the connected regions in $E \backslash \left(\bigcup_{\alpha\in\Phi} H_\alpha\right)$. Therefore, each regular $\gamma \in E$ belongs to some Weyl chamber and we denote this Weyl chamber by $\mathfrak{C}(\gamma)$. As the Weyl chambers are separated by hyperplanes, any two distinct regular vectors γ_1, γ_2 in the same Weyl chamber must both be on the same side of each Hyperplane.

For any $\gamma \in E$ we let $\Phi^+(\gamma) := \{\alpha \in \Phi : (\gamma, \alpha) > 0\}$. Therefore, any two distinct regular vectors γ_1, γ_2 are in the same Weyl chamber if and only if $\Phi^+(\gamma_1) = \Phi^+(\gamma_2)$. If we fix a regular $\gamma \in E$ then a root $\alpha \in \Phi^+(\gamma)$ is called indecomposable if there does not exist other roots $\beta_1, \beta_2 \in \Phi^+(\gamma)$ such that $\alpha = \beta_1 + \beta_2$, otherwise α is called decomposable. We let the set of all indecomposable roots in $\Phi^+(\gamma)$ be denoted by $\Delta(\gamma)$. The set $\Delta(\gamma)$ is a base of Φ. Moreover, if Δ is a base of Φ then there exists a regular $\gamma \in E$ such that $\Delta = \Delta(\gamma)$, a proof can be found in [22]. If Δ is a base of Φ and $\Delta = \Delta(\gamma)$, where γ is regular, then we let $\mathfrak{C}(\gamma)$ be the fundamental Weyl chamber relative to Δ. For example, in the following diagram the dashed lines depict the hyperplanes in the root system of Example 2.1.3 and the shaded region is the fundamental Weyl chamber relative to the base $\{\alpha, \beta\}$:



Given a choice of simple roots $\alpha_1, \ldots, \alpha_\ell$ of a root system Φ, the corresponding *fundamental weights* $\omega_1, \ldots, \omega_\ell$ are defined by

$$\langle \omega_i, \alpha_j \rangle = \delta_{ij}.$$

Moreover, the integer span of the fundamental weights is called to weight lattice, which is denoted by P. The cone $P^+ \subseteq P$ defined by

$$P^+ = \left\{ \sum_{i=1}^{\ell} k_i \omega_i \in P : k_i \geq 0 \text{ for all } i \right\}$$

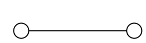
is known as the set of dominant (integral) weights. One particularly important element of P^+ is the *Weyl vector* ρ defined as half sum of the positive roots, i.e.,

$$\rho = \frac{1}{2} \sum_{\alpha\in\Phi^+} \alpha = \sum_{i=1}^{\ell} \omega_i.$$

2.2 Classification of Irreducible Root Systems

In this section we will give a classification of the *irreducible* root systems. A root system Φ is irreducible if cannot be partitioned into the union of two proper subsets $\Phi_1, \Phi_2 \subseteq \Phi$ such that $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. We will achieve this by examining *Dynkin diagrams*, which are named after Eugene Dynkin. We will see that every irreducible root system corresponds to a Dynkin diagram. Before introducing Dynkin diagrams we will first discuss *Coxeter graphs*.

Let Φ be a root system of rank ℓ and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$ be a base of Φ. The Coxeter graph of Φ is the graph with ℓ vertices where the ith and jth vertices, with $i \neq j$, are connected by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges. For example, the root system given in Example 2.1.3 has the Coxeter graph:



The condition (R4), from the definition of root systems, significantly restricts the number of possibilities for the value of $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ for roots $\alpha, \beta \in \Phi$. Recall that if $\alpha, \beta \in E$ then the angle θ between them is given by $\|\alpha\| \cdot \|\beta\| \cos(\theta) = (\alpha, \beta)$ where $\|\alpha\| := (\alpha, \alpha)^{\frac{1}{2}}$ is the *length* of α. Therefore,

$$\langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \frac{2\|\beta\| \cdot \|\alpha\| \cos(\theta)}{\|\alpha\| \cdot \|\alpha\| \cos(0)} = 2\frac{\|\beta\|}{\|\alpha\|} \cos(\theta)$$

and it follows that

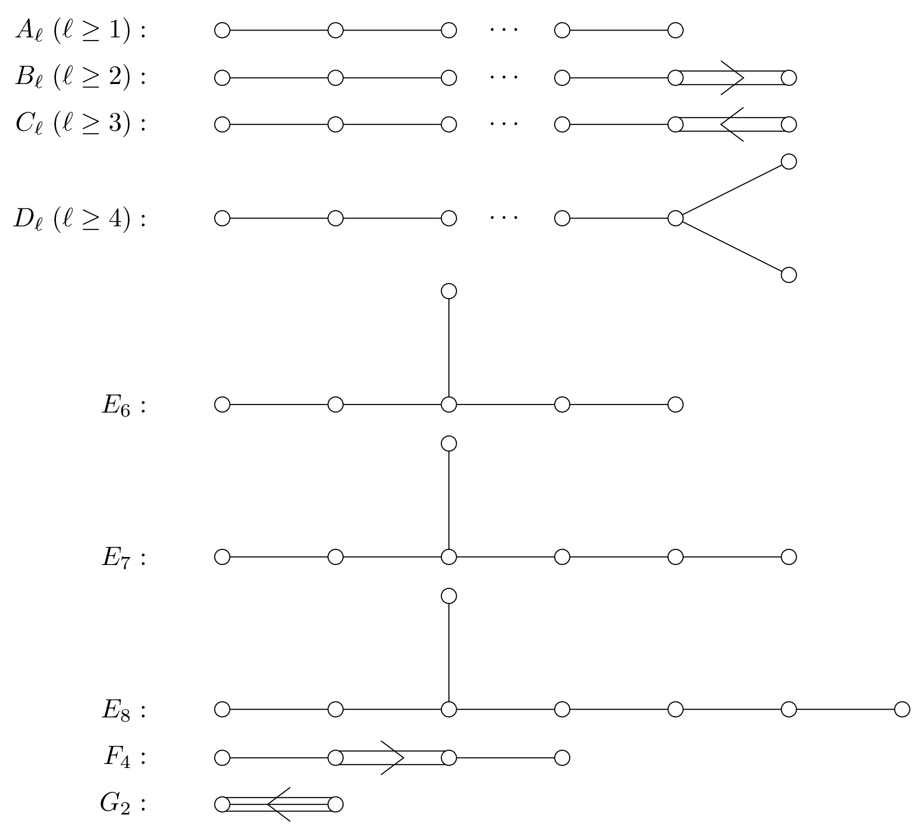
$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4\cos^2(\theta) = \big(2\cos(\theta)\big)^2. \tag{2.2}$$

By (R4) the left-hand side of (2.2) must be an integer. This leaves the following possibilities

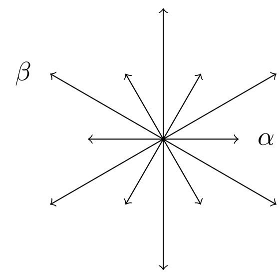
$$\cos(\theta) \in \big\{0, \pm\tfrac{1}{2}, \pm\tfrac{\sqrt{2}}{2}, \pm\tfrac{\sqrt{3}}{2}, \pm 1\big\}.$$

Therefore, $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$ is equal to 0, 1, 2, 3 or 4. If α_i and α_j are distinct simple roots then they cannot be scalar multiples of each other, which implies $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle \neq 4$. Therefore, in the Coxeter graph of Φ the ith and jth vertices, with $i \neq j$, will be connected by 0, 1, 2 or 3 edges.

A Dynkin diagram of Φ is formed by taking the Coxeter digram of Φ and whenever two vertices are connected by 2 or 3 edges we add an arrow pointing to the *shorter* of the two roots, i.e., the root with least length. If Φ is an irreducible root system then at most two root lengths occur in Φ, [22]. Thus, the roots with the longest length we call *long roots* and the roots with the shortest length we call *short roots* (if all the roots have the same length then we call them all long roots). If Φ is an irreducible root system of rank ℓ then its Dynkin diagram is one of the following (the subscripts denote the number of vertices, i.e., the rank of the root system):



A proof of this classification can be found in [22]. The root system in $\mathbb{R}^2$ with base $\{\alpha, \beta\}$ depicted below:



has the Dynkin diagram G_2.

2.3 The Weyl Character Formula

In the representation theory of semi-simple Lie algebras, a particularly important role is played by the so-called integrable highest-weight representations. Let $\mathfrak{g}$ be a semi-simple Lie algebra with root system Φ. For each $\lambda \in P^+$ there is, up to isomorphism, exactly one irreducible highest-weight representation of $\mathfrak{g}$, denoted $V(\lambda)$, [22]. The character of $V(\lambda)$ can be expressed concisely using only root system data, thanks to the *Weyl character formula*:

$$\operatorname{char}\big(V(\lambda)\big) = \frac{\sum_{w\in\mathscr{W}} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\succ 0}(1-e^{-\alpha})}.$$

Here ρ is the Weyl vector and $\mathscr{W}$ the Weyl group. The Weyl character formula is named after Hermann Weyl. He gives and proves this formula in the sequence of three papers [23] [24] [25]. For $\lambda = 0$ we obtain the one-dimensional trivial representation, in which case the character trivialises to 1. In this case we obtain the *Weyl denominator formula*:

$$\sum_{w\in\mathscr{W}} (-1)^{\ell(w)} e^{w(\rho)-\rho} = \prod_{\alpha\succ 0}(1-e^{-\alpha}).$$

Of particular interest to us are the Lie algebras $\mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{so}(2n+1,\mathbb{C})$ whose root systems are A_{n-1} and B_n respectively. For these two cases the above characters may be identified with the Schur functions (1.5) and *odd-orthogonal Schur functions*

$$\mathrm{so}_{2n+1,\lambda}(x) := \frac{1}{\Delta(x)} \det_{1\le i,j\le n}\big(x_i^{-\lambda_j+j-1} - x_i^{\lambda_j+2n-j}\big),$$

where $x=(x_1,\dots,x_n)$, $\lambda=(\lambda_1,\dots,\lambda_n)$ is a partition or '*half-partition*' (a half-partition is where each $\lambda_i \in \mathbb{Z}+\frac{1}{2}$) and

$$\Delta(x) := \det_{1\le i,j\le n}(x_i^{j-1} - x_i^{2n-j}) = \prod_{i=1}^{n}(1-x_i) \prod_{1\le i<j\le n}(x_i-x_j)(x_ix_j-1).$$

To make the above connection in the case of B_n one takes $E=\mathbb{R}^n$, with orthonormal standard basis vectors $\epsilon_1,\dots,\epsilon_n$, [7]. Then we let $\Phi=\Phi^+\cup\Phi^-$ be the root system of rank n where

$$\Phi^+ = \{\epsilon_i \pm \epsilon_j : 1\le i<j\le n\} \cup \{\epsilon_i : 1\le i\le n\}, \qquad \Phi^- = \{-\epsilon_i \pm \epsilon_j : 1\le i<j\le n\} \cup \{-\epsilon_i : 1\le i\le n\}.$$

It has base

$$\Delta = \{\epsilon_1-\epsilon_2,\dots,\epsilon_{n-1}-\epsilon_n,\epsilon_n\}$$

and the Weyl vector is

$$\rho = \frac{1}{2}\big((2n-1)\epsilon_1 + (2n-3)\epsilon_2 + \cdots + \epsilon_n\big).$$

The set of dominant integral weights may be parametrised as

$$(\lambda_1-\lambda_2)\omega_1 + \cdots + (\lambda_{n-1}-\lambda_n)\omega_{n-1} + 2\lambda_n\omega_n.$$

One can identify the Weyl group of this root system as the group of signed permutations, i.e., the hyperoctahedral group $(\mathbb{Z}/2\mathbb{Z})\wr S_n$, where S_n acts on $\mathbb{R}^n$ by permuting the ϵ_i's and $\mathbb{Z}/2\mathbb{Z}$ by negating the ϵ_i's. A similar construction of type A_{n-1} expresses the character, in the Weyl character formula, as the Schur function (1.5).

2.4 Proof of Theorem 1.2.3

In this Section we will give an overview of Macdonald's proof of Theorem 1.2.3 which he gives in [7]. Macdonald was able to discover a proof for Theorem 1.2.3 using results from representation theory, in particular the Weyl character formula. Let r and t be non-negative integers. In [7] Macdonald derived the following identity:

$$\sum_{\lambda \subseteq (t^r)} s_\lambda(x_1, \ldots, x_r) = \frac{\det_{1\le i,j\le r}\left(x_i^{j-1} - x_i^{t+2r-j}\right)}{\prod_{i=1}^{r}(1-x_i)\prod_{1\le i<j\le r}(x_i - x_j)(x_i x_j - 1)}. \tag{2.3}$$

This may be recognised, see [7], as the branching rule:

$$\sum_{\lambda \subseteq (t^r)} s_\lambda(x_1, \ldots, x_r) = (x_1 \cdots x_n)^{\frac{t}{2}} \ \mathrm{so}_{2r+1,(\frac{t}{2})^r}(x_1, \ldots, x_r).$$

We now specialise by setting $x_i = q^{2r-2i+1}$ for each $i = 1, 2, \ldots, r$. In the denominator of (2.3) we obtain

$$\prod_{i=1}^{r}\left(1 - q^{2r-2i+1}\right) \prod_{1\le i<j\le r}\left(q^{2r-2i+1} - q^{2r-2j+1}\right)\left(q^{2r-2i+1}q^{2r-2j+1} - 1\right). \tag{2.4}$$

Replacing i and j with $r-i+1$ and $r-j+1$, respectively, and swapping i and j in the inequality of the right product (as $i<j$ implies $r-j+1<r-i+1$) it follows that (2.4) is equal to

$$\begin{aligned}
&\prod_{i=1}^{r}\left(1 - q^{2r-2(r-i+1)+1}\right) \prod_{1\le j<i\le r}\left(q^{2r-2(r-i+1)+1} - q^{2r-2(r-j+1)+1}\right)\left(q^{2r-2(r-i+1)+1}q^{2r-2(r-j+1)+1} - 1\right)\\
&= \prod_{i=1}^{r}\left(1 - q^{2i-1}\right) \prod_{1\le j<i\le r}\left(q^{2i-1} - q^{2j-1}\right)\left(q^{2i-1}q^{2j-1} - 1\right)\\
&= \prod_{i=1}^{r}\left(1 - q^{2i-1}\right) \prod_{1\le j<i\le r}\Big((1 - q^{2j+2i-2})(1 - q^{2i-2j})q^{2j-1}\Big).
\end{aligned}$$

Relabelling i to j and j to i in the right product implies that (2.4) is equal to

$$\prod_{i=1}^{r}\left(1 - q^{2i-1}\right) \prod_{1\le i<j\le r}\Big((1 - q^{2i+2j-2})(1 - q^{2j-2i})q^{2i-1}\Big). \tag{2.5}$$

For the numerator of (2.3) by a non-trivial application of the B_r Weyl denominator formula, see [26] for details, we obtain

$$\det_{1\le i,j\le r}\left(\left(q^{2r-2i+1}\right)^{t+2r-j} - \left(q^{2r-2i+1}\right)^{j-1}\right) = \prod_{i=1}^{r}\left(1 - q^{t+2i-1}\right) \prod_{1\le i<j\le r}\Big((1 - q^{2t+2i+2j-2})(1 - q^{2j-2i})q^{2i-1}\Big). \tag{2.6}$$

It follows from (2.3), (2.5) and (2.6) that

$$\begin{aligned}
\sum_{\lambda \subseteq (t^r)} s_\lambda(q^{2r-1}, q^{2r-3}, \ldots, q) &= \frac{\prod_{i=1}^{r}\left(1 - q^{t+2i-1}\right)\prod_{1\le i<j\le r}\Big((1 - q^{2t+2i+2j-2})(1 - q^{2j-2i})q^{2i-1}\Big)}{\prod_{i=1}^{r}\left(1 - q^{2i-1}\right)\prod_{1\le i<j\le r}\Big((1 - q^{2i+2j-2})(1 - q^{2j-2i})q^{2i-1}\Big)}\\
&= \prod_{i=1}^{r}\frac{1 - q^{t+2i-1}}{1 - q^{2i-1}} \prod_{1\le i<j\le r}\frac{1 - q^{2(t+i+j-1)}}{1 - q^{2(i+j-1)}}
\end{aligned}$$

where, by Theorem 1.3.7, the left-hand side is the generating function for symmetric plane partitions contained in $\mathcal{B}(r, r, t)$. This concludes the proof of Theorem 1.2.3.

Chapter 3

Fock Space and Vertex Operators

In this chapter we will examine operators on the fermionic Fock space and use these to study plane partitions and symmetric functions. The fermionic Fock space is named after the physicist Vladimir A. Fock (1898–1974) whose work helped develop the field of quantum physics and quantum mechanics. This space was first introduced by Fock in 1932 [27]. The fermionic Fock space is an algebraic construction (which is given in Section 3.1) that is used to describe quantum states. It is related to the Dirac sea, which was introduced by the physicist Paul Dirac in 1930 [28]. The Dirac sea is related to the energy in a quantum state. An algebraic interpretation of this is given in Section 3.2. In Section 3.3 we will examine the relationship between the Dirac sea and Maya diagrams. This will result in a connection between partitions and zero-charge quantum states. In Section 3.4 we will study vertex operators on the fermionic Fock space. We will use the results of Section 3.4 in Section 3.5 to compute generating functions for plane partitions. For more information about the fermionic Fock space and vertex operators one can see [8] [9] [10] [29].

3.1 Fock Space

In this section we will give an algebraic construction of the fermionic Fock space. For this we must first introduce the exterior algebra of a vector space W, denoted $\Lambda(W)$. It is defined as the quotient $T(W)/I$ where $T(W)$ is the tensor algebra on W

$$T(W) = \bigoplus_{k=0}^{\infty} T^k W, \qquad T^k W = \bigotimes_{j=1}^{k} W$$

and I is the ideal generated by all $v \otimes v$ where $v \in W$. The wedge product $\wedge$ is defined by $v \wedge w = v \otimes w + I$ for any $v, w \in W$. The wedge product is associative and distributive. From the definition of I it follows that $v \wedge v = 0$ for all $v \in W$. The wedge product is anticommutative since

$$0 = (v + w) \wedge (v + w) = v \wedge v + v \wedge w + w \wedge v + w \wedge w = v \wedge w + w \wedge v$$

which means $v \wedge w = -(w \wedge v)$ for all $v, w \in W$.

From now on V is the infinite-dimensional vector space over $\mathbb{F}$ with basis $\{v_k\}_{k \in \mathbb{Z}+\frac{1}{2}}$. The use of half integers for indexing basis elements of V stems from the concept of spin in quantum mechanics. Specifically, fermions are particles with half-integer spins. For example, the wedge product

$$- v_{\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{7}{2}} \wedge v_{\frac{5}{2}} \wedge v_{-\frac{9}{2}} \wedge v_{-\frac{11}{2}} \wedge \cdots \tag{3.1}$$

is an element of the completion of $\Lambda(V)$. By the anticommutativity of the wedge product, (3.1) can be reordered so that vector subscripts are strictly decreasing. Namely, it is equal to

$$v_{\frac{5}{2}} \wedge v_{\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{7}{2}} \wedge v_{-\frac{9}{2}} \wedge v_{-\frac{11}{2}} \wedge \cdots . \tag{3.2}$$

A wedge product $\bigwedge_{i=1}^{\infty} v_{s_i} = v_{s_1} \wedge v_{s_2} \wedge \cdots$ in the completion of $\Lambda(V)$ is called *semi-infinite* if the s_i's are pairwise distinct and the set $\{s_1, s_2, \ldots\}$ is bounded from above [30]. A semi-infinite wedge product $\bigwedge_{i=1}^{\infty} v_{s_i}$ is called *normally ordered* if its vector subscripts are strictly decreasing, i.e., $s_1 > s_2 > \cdots$. A semi-infinite wedge product $\bigwedge_{i=1}^{\infty} v_{s_i}$ is called *regular* if there exists some integer n such that $s_i - s_{i+1} = 1$ for all $i > n$. The example (3.2) is a regular normally-ordered semi-infinite wedge product. Given any semi-infinite wedge product it can be normally ordered using the anticommutativity of the wedge product.

If $S = \{s_1, s_2, \dots\} \subset \mathbb{Z} + \frac{1}{2}$ is an ordered set then we let $v_S := \bigwedge_{i=1}^{\infty} v_{s_i}$. For example, if

$$S = \left\{\tfrac{5}{2}, \tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{7}{2}, -\tfrac{9}{2}, -\tfrac{11}{2}, \dots\right\} \tag{3.3}$$

then v_S is given by (3.2). If $S \subset \mathbb{Z} + \frac{1}{2}$ is an ordered set such that its elements are strictly decreasing then v_S is a regular normally-ordered semi-infinite wedge product if and only if the sets

$$S_+ := S \backslash \left\{-\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}, \dots\right\}, \quad S_- := \left\{-\tfrac{1}{2}, -\tfrac{3}{2}, -\tfrac{5}{2}, \cdots\right\} \backslash S$$

are finite. The set S_+ is finite if and only if S has an upper bound, i.e., v_S is semi-infinite. The set S_- is finite if and only if there exists some positive integer n such that $s_i - s_{i+1} = 1$ for all $i > n$, that is v_S is regular. For example, for S in (3.3) we have $S_+ = \{\frac{5}{2}, \frac{1}{2}\}$ and $S_- = \{-\frac{1}{2}, -\frac{5}{2}\}$.

We denote the fermionic Fock space (also known as the semi-infinite wedge space) as $\Lambda^{\frac{\infty}{2}} V$. It is the subspace of the completion of $\Lambda(V)$ with basis the set $\{v_S\}$ of regular normally-ordered semi-infinite wedge products [8]. We refer to the elements of the fermionic Fock space as *states*. For example, for S in (3.3) the state v_S is a state in the basis $\{v_S\}$ of $\Lambda^{\frac{\infty}{2}} V$. Whenever we write v_S this is assumed to be an element of the basis $\{v_S\}$. Given the state v_S we define $s_0 = \infty$ which is not in S. We equip $\Lambda^{\frac{\infty}{2}} V$ with an inner product $(\cdot, \cdot)$ in which the basis $\{v_S\}$ is orthonormal, i.e., $(v_S, v_{S'}) = \delta_{S,S'}$ where $\delta_{S,S'}$ is the Kronecker delta

$$\delta_{S,S'} := \begin{cases} 1, & \text{if } S = S' \\ 0, & \text{otherwise.} \end{cases}$$

In the Dirac sea, a quantum state can correspond to a state in the fermionic Fock space, [31]. An important state in the fermionic Fock space is the *vacuum state* defined as

$$v_0 := v_{-\frac{1}{2}} \wedge v_{-\frac{3}{2}} \wedge v_{-\frac{5}{2}} \wedge \cdots . \tag{3.4}$$

The charge of a state v_S is the integer $|S_+| - |S_-|$. For example, the vacuum state v_0 has zero charge. We denote by F_n the subspace of $\Lambda^{\frac{\infty}{2}} V$ spanned by all regular normally-ordered semi-infinite wedge products of charge n. The charge provides a $\mathbb{Z}$-grading of the fermionic Fock space, namely

$$\Lambda^{\frac{\infty}{2}} V = \bigoplus_{n \in \mathbb{Z}} F_n. \tag{3.5}$$

In Section 3.2 we see how to give (3.5) in terms of the zero charge space F_0. The space F_0 is of particular interest as in Section 3.3 we will see a bijection between partitions and regular normally-ordered semi-infinite wedge products of zero charge.

3.2 Creation and Annihilation Operators

We will now introduce two important operators that act on the fermionic Fock space. These operators are called *creation* and *annihilation* operators. In quantum mechanics the creation and annihilation operators increase and decrease, respectively, the number of particles in a quantum state by one [32]. Let $\psi_k : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ be a linear operator defined by $\psi_k(f) = v_k \wedge f$ where $k \in \mathbb{Z} + \frac{1}{2}$ and v_k is a vector in the basis $\{v_k\}_{k \in \mathbb{Z} + \frac{1}{2}}$ of V. The operator ψ_k acts on the state v_S (recall $s_0 := \infty$) as follows:

$$\psi_k(v_S) = \begin{cases} (-1)^r v_{s_1} \wedge \cdots \wedge v_{s_r} \wedge v_k \wedge v_{s_{r+1}} \wedge \cdots , & \text{if } s_r > k > s_{r+1} \\ 0, & \text{if } k \in S \end{cases}$$

which, up to a possible sign, is a regular normally-ordered semi-infinite wedge product in the $k \notin S$ case. We call ψ_k a creation operator. This operator corresponds to the action of creating a fermion in a quantum state if that fermion does not already belong to that state. If a fermion exists in a quantum state, then we cannot create another identical fermion in the state by the Pauli exclusion principle [33].

Let $\psi_k^* : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$, where $k \in \mathbb{Z} + \frac{1}{2}$, be the adjoint operator of ψ_k, i.e., $\big(\psi_k(f), g\big) = \big(f, \psi_k^*(g)\big)$ for all states $f, g \in \Lambda^{\frac{\infty}{2}} V$. In the following Lemma we will see explicitly how ψ_k^* acts on the fermionic Fock space.

Lemma 3.2.1. *The operator ψ_k^* acts on a state v_S as follows:*

$$\psi_k^*(v_S) = \begin{cases} (-1)^{r-1} v_{s_1} \wedge \cdots \wedge v_{s_{r-1}} \wedge v_{s_{r+1}} \wedge \cdots, & \text{if } k = s_r \\ 0, & \text{if } k \notin S. \end{cases} \tag{3.6}$$

which, up to a sign, is a regular normally-ordered semi-infinite wedge product in the $k \in S$ case.

Proof. We need to show (3.6) satisfies the relation $\big(\psi_k(f), g\big) = \big(f, \psi_k^*(g)\big)$ for all states f and g in the basis $\{v_S\}$ of $\Lambda^{\frac{\infty}{2}} V$. If f and g are states in the basis $\{v_S\}$ then $f = v_I$ and $g = v_J$ for some ordered sets $I = \{i_1, i_2, \ldots\} \subset \mathbb{Z} + \frac{1}{2}$ and $J = \{j_1, j_2, \ldots\} \subset \mathbb{Z} + \frac{1}{2}$ such that $i_2 > i_2 > \cdots$ and $j_1 > j_2 > \cdots$ with the sets I_+, I_-, J_+ and J_- being finite. We have the following cases:

(1) If $k \in I$ then $\psi_k(f) = 0$ and $f \neq \psi_k^*(g)$, which implies $\big(\psi_k(f), g\big) = 0$ and $\big(f, \psi_k^*(g)\big) = 0$.

(2) Suppose $k \notin I$.

  (a) Suppose $I \cup k \neq J$. Then $\big(\psi_k(f), g\big) = 0$ and either $\psi_k^*(g) = 0$ (if $k \notin J$) or $I \neq J \backslash \{k\}$ (if $k \in J$), with either implying $\big(f, \psi_k^*(g)\big) = 0$.

  (b) If $I \cup k = J$ then $k = j_r$ for some r. This implies $\psi_k(f) = (-1)^{r-1} g$ and $\psi_k^*(g) = (-1)^{r-1} f$ which gives $\big(\psi_k(f), g\big) = (-1)^{r-1} = \big(f, \psi_k^*(g)\big)$.

We conclude that $\big(\psi_k(f), g\big) = \big(f, \psi_k^*(g)\big)$ for arbitrary states f and g in the basis $\{v_S\}$ of $\Lambda^{\frac{\infty}{2}} V$. Therefore, ψ_k^* defined in (3.6) is the adjoint operator of ψ_k. □

The equation (3.6) extends to all states in $\Lambda^{\frac{\infty}{2}} V$ by linearity. We call ψ_k^* an annihilation operator. This operator corresponds to the action of removing a fermion from a quantum state, providing that fermion does exist in that state.

Lemma 3.2.2. *The operators ψ_k and ψ_k^* satisfy the anticommutation relation $\psi_k \psi_k^* + \psi_k^* \psi_k = 1$.*

Proof. Let $f = v_S$ be a state in the basis $\{v_S\}$ of $\Lambda^{\frac{\infty}{2}} V$. We have the following two cases:

(1) Suppose $k \in S$. Then $k = s_r$ for some r. It follows that

$$\begin{aligned} \psi_k\big(\psi_k^*(f)\big) &= \psi_k\big((-1)^{r-1} v_k \wedge v_{s_1} \wedge \cdots \wedge v_{s_{r-1}} \wedge v_{s_{r+1}} \wedge \cdots\big) \\ &= (-1)^{r-1}(-1)^{r-1} v_{s_1} \wedge \cdots \wedge v_{s_{r-1}} \wedge v_{s_r} \wedge v_{s_{r+1}} \wedge \cdots \\ &= f. \end{aligned}$$

  Moreover, $\psi_k(f) = 0$ which means $\psi_k^*\big(\psi_k(f)\big) = 0$. Therefore, we have $(\psi_k \psi_k^* + \psi_k^* \psi_k)(f) = f$.

(2) Suppose $k \notin S$. Then there exists some r such that $s_r > k > s_{r+1}$. This implies

$$\begin{aligned} \psi_k^*\big(\psi_k(f)\big) &= \psi_k^*\big((-1)^{r} v_{s_1} \wedge \cdots \wedge v_{s_r} \wedge v_k \wedge v_{s_{r+1}} \wedge \cdots\big) \\ &= (-1)^{r}(-1)^{r} v_{s_1} \wedge \cdots \wedge v_{s_{r-1}} \wedge v_{s_r} \wedge v_{s_{r+1}} \wedge \cdots \\ &= f. \end{aligned}$$

  Moreover, $\psi_k^*(f) = 0$ which means $\psi_k\big(\psi_k^*(f)\big) = 0$. Therefore, we have $(\psi_k \psi_k^* + \psi_k^* \psi_k)(f) = f$.

Therefore, in either case $(\psi_k \psi_k^* + \psi_k^* \psi_k)(f) = f$ where f is an arbitrary state in the basis $\{v_S\}$ of $\Lambda^{\frac{\infty}{2}} V$. This implies the anticommutation relation $\psi_k \psi_k^* + \psi_k^* \psi_k = 1$. □

In general, the creation and annihilation operators satisfy the anticommutation relations

$$\psi_i \psi_j^* + \psi_i^* \psi_j = \delta_{i,j}, \tag{3.7a}$$
$$\psi_i \psi_j + \psi_j \psi_i = 0, \tag{3.7b}$$
$$\psi_i^* \psi_j^* + \psi_j^* \psi_i^* = 0 \tag{3.7c}$$

where $\delta_{i,j}$ is the Kronecker delta. Lemma 3.2.2 proves the non-trivial anticommutation relation in (3.7). The proof of the other relations in (3.7) follow from straightforward case checking. The infinite Clifford algebra is the algebra generated by the operators ψ_k, ψ_k^* (where k runs over all half integers) subject to the relations (3.7).

It follows from Lemma 3.2.2 that the operators ψ_k and ψ_k^* satisfy the relations

$$\psi_k\psi_k^*(v_S) = \begin{cases} v_S, & \text{if } k \in S \\ 0, & \text{if } k \notin S, \end{cases} \qquad \psi_k^*\psi_k(v_S) = \begin{cases} 0, & \text{if } k \in S \\ v_S, & \text{if } k \notin S. \end{cases} \tag{3.8}$$

We define the *charge operator* $C : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ as

$$C = \sum_{k \in \mathbb{Z}+\frac{1}{2}} : \psi_k\psi_k^* : \tag{3.9}$$

where : : denotes normal ordering, i.e.,

$$: \psi_k\psi_k^* : = \begin{cases} \psi_k\psi_k^*, & \text{if } k > 0 \\ -\psi_k^*\psi_k & \text{if } k < 0. \end{cases}$$

From now on when we write an operator acting on a state we will often drop the parentheses to avoid excessive notation, for example $\psi_k v_S = \psi_k(v_S)$. In the following Lemma we will see how the charge operator gains its name.

Lemma 3.2.3. *We have the following relation:*

$$Cv_S = (|S_+| - |S_-|)v_S. \tag{3.10}$$

Proof. We first split the sum of C into two cases

$$Cv_S = \sum_{k \in \mathbb{Z}+\frac{1}{2}} : \psi_k\psi_k^* : v_S = \sum_{\substack{k \in \mathbb{Z}+\frac{1}{2} \\ k>0}} : \psi_k\psi_k^* : v_S + \sum_{\substack{k \in \mathbb{Z}+\frac{1}{2} \\ k<0}} : \psi_k\psi_k^* : v_S = \sum_{\substack{k \in \mathbb{Z}+\frac{1}{2} \\ k>0}} \psi_k\psi_k^* v_S - \sum_{\substack{k \in \mathbb{Z}+\frac{1}{2} \\ k<0}} \psi_k^*\psi_k v_S.$$

It follows from (3.8) that

$$Cv_S = \sum_{k \in S_+} v_S - \sum_{k \in S_-} v_S = (|S_+| - |S_-|)v_S. \qquad \square$$

For example, for S in (3.3) we have $Cv_S = 0$. It follows from Lemma 3.2.3 that the space F_0 is the kernel of the charge operator C. We will now see how to give the decomposition (3.5) in terms of the space F_0 together with the translation operator $R : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$, which is defined by

$$Rv_{s_1} \wedge v_{s_2} \wedge \cdots = v_{s_1+1} \wedge v_{s_2+1} \wedge \cdots$$

with its inverse R^{-1} defined by $R^{-1}v_{s_1} \wedge v_{s_2} \wedge \cdots = v_{s_1-1} \wedge v_{s_2-1} \wedge \cdots$. For example, for S in (3.3) we have

$$R^n v_S = v_{\frac{5}{2}+n} \wedge v_{\frac{1}{2}+n} \wedge v_{-\frac{3}{2}+n} \wedge v_{-\frac{7}{2}+n} \wedge v_{-\frac{9}{2}+n} \wedge v_{-\frac{11}{2}+n} \wedge \cdots$$

for any integer n. We call $R^n v_0$ the vacuum state in the space F_n.

Lemma 3.2.4. *The operators ψ_k, ψ_k^* and R satisfy the relations $R\psi_k R^{-1} = \psi_{k+1}$ and $R\psi_k^* R^{-1} = \psi_{k+1}^*$.*

Proof. We have the following cases:

(1) If $k+1 \in S$ then $k+1 = s_r$ for some r which implies

$$\begin{aligned} R\psi_k^* R^{-1} v_S &= R\psi_k^*(v_{s_1-1} \wedge v_{s_2-1} \wedge \cdots \wedge v_{s_{r-1}-1} \wedge v_k \wedge v_{s_{r+1}-1} \wedge \cdots) \\ &= (-1)^{r-1} R(v_{s_1-1} \wedge v_{s_2-1} \wedge \cdots \wedge v_{s_{r-1}-1} \wedge v_{s_{r+1}-1} \wedge \cdots) \\ &= (-1)^{r-1} v_{s_1} \wedge v_{s_2} \wedge \cdots \wedge v_{s_{r-1}} \wedge v_{s_{r+1}} \wedge \cdots \\ &= \psi_{k+1}^* v_S \end{aligned}$$

and $R\psi_k R^{-1} v_S = R\psi_k(v_{s_1-1} \wedge v_{s_2-1} \wedge \cdots \wedge v_{s_{r-1}-1} \wedge v_k \wedge v_{s_{r+1}-1} \wedge \cdots) = 0 = \psi_{k+1} v_S$.

(2) If $k+1 \notin S$ then $s_r > k+1 > s_{r+1}$ for some r and

$$\begin{aligned} R\psi_k R^{-1} v_S &= R\psi_k(v_{s_1-1} \wedge v_{s_2-1} \wedge \cdots \wedge v_{s_r-1} \wedge v_{s_{r+1}-1} \wedge \cdots) \\ &= (-1)^r R(v_{s_1-1} \wedge v_{s_2-1} \wedge \cdots \wedge v_{s_r-1} \wedge v_k \wedge v_{s_{r+1}-1} \wedge \cdots) \\ &= (-1)^r v_{s_1} \wedge v_{s_2} \wedge \cdots \wedge v_{s_r} \wedge v_{k+1} \wedge v_{s_{r+1}} \wedge \cdots \\ &= \psi_{k+1} v_S. \end{aligned}$$

Also, $R\psi_k^* R^{-1} v_S = 0 = \psi_{k+1}^* v_S$.

As v_S is an arbitrary state in the basis $\{v_S\}$ it follows that $R\psi_k R^{-1} = \psi_{k+1}$ and $R\psi_k^* R^{-1} = \psi_{k+1}^*$. $\square$

Therefore, $\psi_{k+1}\psi_{k+1}^* = R\psi_k R^{-1} R\psi_k^* R^{-1} = R\psi_k\psi_k^* R^{-1}$ and $\psi_{k+1}^*\psi_{k+1} = R\psi_k^*\psi_k R^{-1}$. This implies

$$\psi_k\psi_k^* = R^n\psi_{k-n}\psi_{k-n}^* R^{-n}, \qquad \psi_k^*\psi_k = R^n\psi_{k-n}^*\psi_{k-n}R^{-n} \tag{3.11}$$

for all integers n. We will use these relations to prove the following Lemma.

Lemma 3.2.5. *The charge operator satisfies the relation*

$$C = R^n(C+n)R^{-n} \tag{3.12}$$

for all integers n.

Proof. By (3.11) we have

$$C = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_k^* - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_k^*\psi_k = R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi_{k-n}^* - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_{k-n}^*\psi_{k-n}\right)R^{-n}$$

for all integers n. Clearly (3.12) holds for $n=0$. If $n>0$ then

$$\begin{aligned}
C &= R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi_{k-n}^* - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_{k-n}^*\psi_{k-n}\right)R^{-n}\\
&= R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>n}} \psi_{k-n}\psi_{k-n}^* - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<n}} \psi_{k-n}^*\psi_{k-n}\right)R^{-n} + R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0<k<n}} (\psi_{k-n}\psi_{k-n}^* + \psi_{k-n}^*\psi_{k-n})\right)R^{-n}\\
&= R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_k^* - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_k^*\psi_k\right)R^{-n} + R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0<k<n}} (\psi_{k-n}\psi_{k-n}^* + \psi_{k-n}^*\psi_{k-n})\right)R^{-n}\\
&= R^n(C+n)R^{-n}
\end{aligned}$$

where the last equality follows from (3.7a). Similarly if $n<0$ then

$$\begin{aligned}
C &= R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>n}} \psi_{k-n}\psi_{k-n}^* - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<n}} \psi_{k-n}^*\psi_{k-n}\right)R^{-n} - R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0>k>n}} (\psi_{k-n}\psi_{k-n}^* + \psi_{k-n}^*\psi_{k-n})\right)R^{-n}\\
&= R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_k^* - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_k^*\psi_k\right)R^{-n} - R^n\left(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0>k>n}} (\psi_{k-n}\psi_{k-n}^* + \psi_{k-n}^*\psi_{k-n})\right)R^{-n}\\
&= R^n(C+n)R^{-n}.
\end{aligned}$$

In conclusion (3.12) holds for all integers n. $\square$

The translation operator maps states in F_n to states in F_{n+1}. This is more formally stated in the following Lemma.

Lemma 3.2.6. *The space $F_n = R^nF_0$ for all $n\in\mathbb{Z}$.*

Proof. We will first show $F_n \subseteq R^nF_0$ and secondly show $F_n \supseteq R^nF_0$.

(1) To show $F_n \subseteq R^nF_0$ let v be an arbitrary state in the basis $\{v_S\}$ such that $v\in F_n$ and let $w = R^{-n}v$. Since $v\in F_n$ then $Cv = nv$ by Lemma 3.2.3. Therefore, $R^{-n}Cv = nR^{-n}v = nw$ and (3.12) implies $nw = R^{-n}Cv = (C+n)R^{-n}v = (C+n)w$. It follows that $Cw = 0$ which implies $w\in F_0$. Therefore, $v = R^nw \in R^nF_0$ which implies $F_n\subseteq R^nF_0$.

(2) To show $R^nF_0 \subseteq F_n$ let v be an arbitrary state in the basis $\{v_S\}$ such that $v \in R^nF_0$. Then $v = R^n w$ for some state $w = R^{-n}v \in F_0$ in the basis $\{v_S\}$. Then $Cw = 0$ which implies $0 = R^nCw = (C-n)R^nw = (C-n)v$. Therefore, $Cv = nv$ and it follows that $v \in F_n$. This implies $R^nF_0 \subseteq F_n$.

In conclusion $F_n = R^nF_0$. □

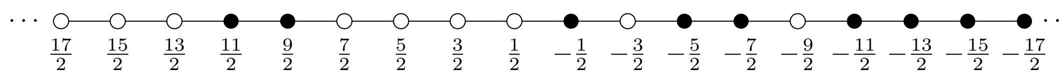
Lemma 3.2.6 and the $\mathbb{Z}$-grading (3.5) allows us to decompose the fermionic Fock space in terms of the translation operator and the space F_0, namely

$$\Lambda^{\frac{\infty}{2}}V = \bigoplus_{n\in\mathbb{Z}} R^nF_0 \tag{3.13}$$

which is called the charge decomposition of $\Lambda^{\frac{\infty}{2}}V$.

3.3 Maya Diagrams

A Maya digram is an assignment of white and black beads to each half integer on the real number line (for Maya diagrams, the number line will have positive numbers to the left of 0), with a finite number of black beads to the left of 0 and a finite number of white beads to the right of 0. For example,



is a Maya diagram. There is a bijection between Maya diagrams and regular normally-ordered semi-infinite wedge products. Namely, given a Maya diagram, let S be the set of half integers corresponding to the black beads in the Maya diagram and then order S so its elements are strictly decreasing. Then S_+ is finite because the number of black beads to the left of 0 is finite. Also S_- is finite because the number of white beads to the right of 0 is finite. It follows that v_S is a regular normally-ordered semi-infinite wedge product. For example, the Maya diagram above gives the state v_S where $S = \{\frac{11}{2}, \frac{9}{2}, -\frac{1}{2}, -\frac{5}{2}, -\frac{7}{2}, -\frac{11}{2}, -\frac{13}{2}, -\frac{15}{2}, -\frac{17}{2}, \ldots\}$. Conversely, given a state v_S, we can create a Maya diagram which has a black bead assigned to each half integer in S and a white bead to all other half integers. This establishes the bijection.

We will now establish a bijection between partitions and zero-charge states in the basis $\{v_S\}$ of $\Lambda^{\frac{\infty}{2}}V$. The bijection is as follows (refer to the diagram below for an example). Let λ be a partition. Take the Young diagram of λ and rotate it by 135 degrees counter clockwise (this is the Russian notation of the Young digram of λ). Then rest it on 0 of the real number line and scale it so that the diagonal slices align with the integers. Extend the leftmost (rightmost) line of the Young diagram indefinitely up and to the left (right). Call the uppermost piecewise-linear curve the contour of the diagram. Assign a white (black) bead to each half integer, on the real line, which is directly below a downwards (upwards) slope on the contour. This will result in a diagram with the following properties:

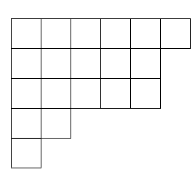
(1) As each part of λ is finite the number of black beads to the left of 0 is finite.

(2) As the number of parts of λ is finite the number of white beads to the right of 0 is finite.

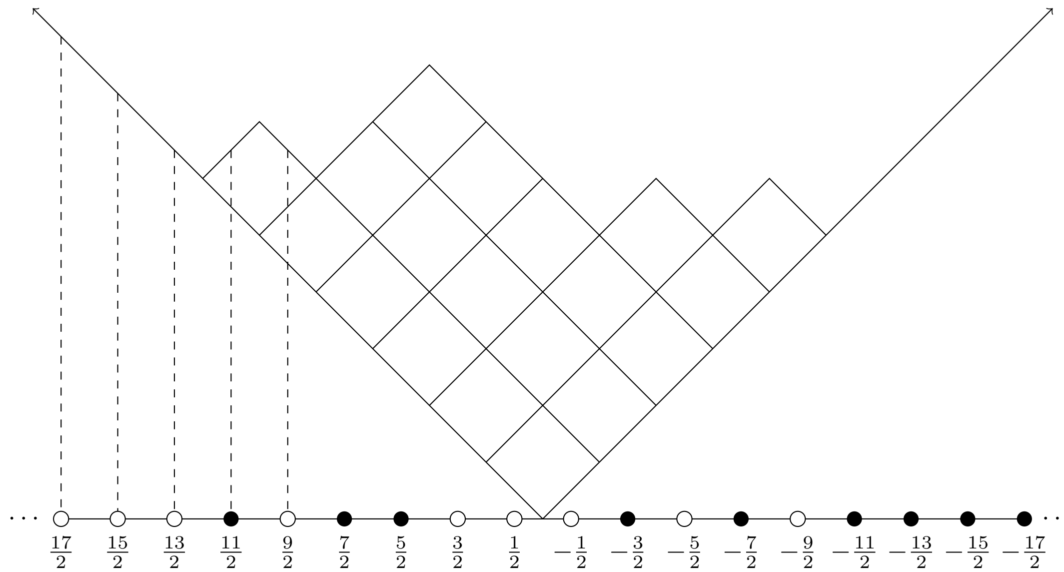
(3) As the main diagonal aligns with 0 the number of black beads to the left of 0 is equal the the number of white beads to the right of 0.

Therefore, a partition λ will give a Maya diagram which corresponds to a regular normally-ordered semi-infinite wedge product of zero charge. This process is easily reversible. Namely, given a regular normally-ordered semi-infinite wedge product of zero charge we can use its corresponding Maya diagram to construct the contour of a rotated Young diagram which gives a partition. This establishes the required bijection.

For example, let $\lambda = (6, 5, 5, 2, 1)$. Then λ corresponds to the Young diagram:



Applying the steps above, we obtain the diagram:



This gives a Maya diagram corresponding to the regular normally-ordered semi-infinite wedge product v_S where

$$S = \{\tfrac{11}{2}, \tfrac{7}{2}, \tfrac{5}{2}, -\tfrac{3}{2}, -\tfrac{7}{2}, -\tfrac{11}{2}, -\tfrac{13}{2}, -\tfrac{15}{2}, -\tfrac{17}{2}, \ldots\}. \tag{3.14}$$

In this example, $|S_+| = |S_-| = 3$ and as expected $v_S \in F_0$. It is easy to generalise the above to a bijection between states in F_n and diagrams shifted a distance n.

We let v_λ denote the state corresponding to a partition λ. Hence by the example above $v_{(6,5,5,2,1)} = v_S$ for S in (3.14). In the bijection between partitions and Maya diagrams described above, the contour line segment at the end of the ith row of the rotated Young diagram of a partition λ will be directly above the half integer $\lambda_i - i + \frac{1}{2}$. Therefore,

$$v_\lambda = \bigwedge_{i=1}^{\infty} v_{\lambda_i - i + \frac{1}{2}}.$$

Moreover, if n is an integer greater than or equal to the length of λ then

$$v_\lambda = \left(\bigwedge_{i=1}^{n} v_{\lambda_i - i + \frac{1}{2}}\right) \wedge R^{-n} v_0.$$

Therefore, we can consider v_λ as the result of taking the state $R^{-n}v_0$ and creating a particle indexed by $\lambda_i - i + \frac{1}{2}$ for each i at most n. That is to say

$$v_\lambda = \left(\prod_{i=1}^{n} \psi_{\lambda_i - i + \frac{1}{2}}\right) R^{-n} v_0. \tag{3.15}$$

The ordering in (3.15) is important as to avoid the sign changing. Moreover, in the bijection above we also obtain

$$\sum_{k \in \mathbb{Z} + \frac{1}{2}} k : \psi_k \psi_k^* : v_\lambda = |\lambda| v_\lambda. \tag{3.16}$$

A Maya diagram gives an illustration of fermionic particles in a quantum state. A black (white) bead on a half integer k indicates that a fermion with spin k exists (does not exist) in the quantum state. With this in mind, we will often refer to the black beads as particles and the white beads as holes.

3.4 Vertex Operators

For any integer n, we define the bosonic operator $\alpha_n : \Lambda^{\frac{\infty}{2}} V \to \Lambda^{\frac{\infty}{2}} V$ as

$$\begin{aligned} \alpha_n &= \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_{k-n} \psi_k^* \quad \text{if } n \neq 0, \\ \alpha_0 &= C \end{aligned}$$

where C is the charge operator (3.9). We will now see that these operators are well defined by discussing how they act on states in the basis $\{v_S\}$ of $\Lambda^{\frac{\infty}{2}}V$. For $n \neq 0$, in the sum

$$\alpha_n v_S = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi_k^* v_S$$

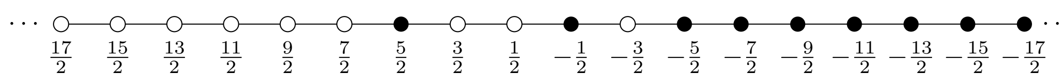
a summand $\psi_{k-n}\psi_k^* v_S$ will be non-zero only when $k \in S$ and $k - n \notin S$. The series $\alpha_n v_S$ will converge to a state in $\Lambda^{\frac{\infty}{2}}V$ as there are only finitely many $k \in S$ such that $k - n \notin S$ by the semi-infinite and regular properties of v_S. Namely, as v_S is semi-infinite there exists a largest half integer $m \in S$ which means $k \notin S$ for all half integers $k > m$. By the regular property of v_S there exists a half integer $c \in S$ such that $k - n \in S$ for all half integers $k < c$. Therefore, the series $\alpha_n v_S$ converges to a state in $\Lambda^{\frac{\infty}{2}}V$ for any state v_S in the basis $\{v_S\}$. This shows that α_n is well defined for $n \neq 0$. We need to define α_n for $n = 0$ differently as the series

$$\sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_k\psi_k^* v_S = \sum_{k \in S} v_S$$

does not converge to a state in $\Lambda^{\frac{\infty}{2}}V$, i.e., letting $\alpha_0 = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_k\psi_k^*$ would not be well defined. Therefore, we define α_0 as the charge operator C which is well defined by Lemma 3.2.3.

We have already seen in Lemma 3.2.3 how α_0 acts on v_S. We will now use Maya diagrams to give a visual description of how α_n acts on v_S for non-zero n. Suppose k is a half integer such that $\psi_{k-n}\psi_k^* v_S$ is non-zero. The operator $\psi_{k-n}\psi_k^*$ corresponds to the action of annihilating the particle indexed by k and then creating a particle indexed by $k - n$ in the quantum state corresponding to v_S. In terms of the Maya diagram of v_S, the operator $\psi_{k-n}\psi_k^*$ corresponds to the action of moving the particle at position k into the hole at position $k - n$. Therefore, $\psi_{k-n}\psi_k^* v_S = \pm v'_S$ where v'_S is the wedge product corresponding to the Maya diagram of v_S in which the particle at position k has moved into the hole at position $k - n$ and the sign is determined by the number of particles jumped over in the process, with positive (negative) sign if an even (odd) number of particles were jumped.

For example, let $n = -3$ and $S = \{\frac{5}{2}, -\frac{1}{2}, -\frac{5}{2}, -\frac{7}{2}, -\frac{9}{2}, \ldots\}$. Then v_S corresponds to the Maya diagram:

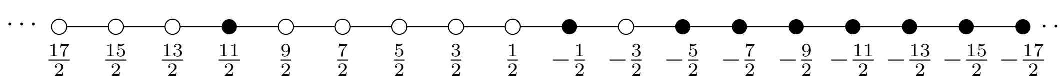


In this Maya diagram a bead at position $k - n$ will be to the left of a bead at position k, because n is negative. There are only three particles that have a hole three places to the left. Namely, the particles indexed by $\frac{5}{2}$, $-\frac{5}{2}$ and $-\frac{9}{2}$. Therefore,

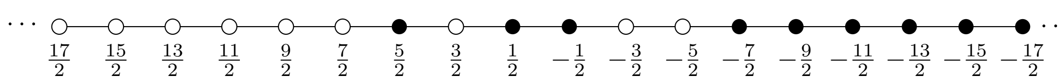
$$\alpha_{-3} v_S = v_1 - v_2 + v_3 \tag{3.17}$$

where v_1, v_2 and v_3 correspond to the following three Maya diagrams:

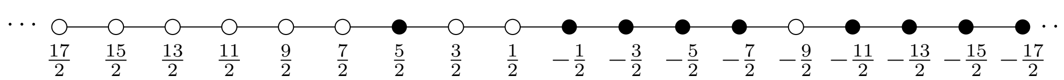
v_1 :



v_2 :



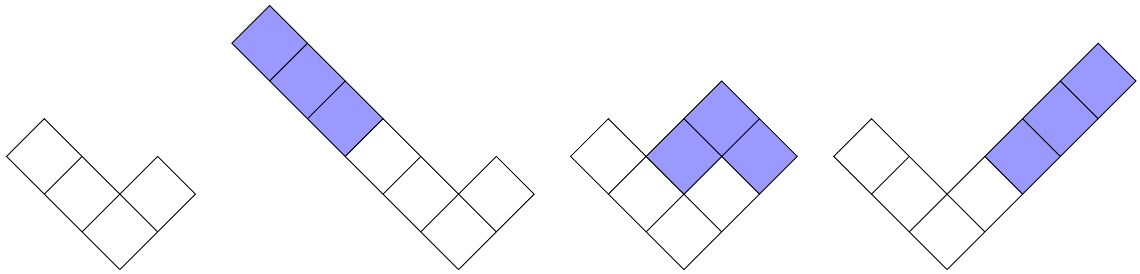
v_3 :



The Maya diagrams of v_1, v_2 and v_3 are the Maya diagram of v_S in which the particle at position $\frac{5}{2}$, $-\frac{5}{2}$ and $-\frac{9}{2}$ are moved three places to the left, respectively. In the Maya digram of v_S, for the particle at position $-\frac{5}{2}$ to move into the hole three places to the left it must jump over the particle at position $-\frac{1}{2}$, thus v_2 has negative sign in (3.17).

The operator $\psi_{k-n}\psi_k^*$ has another combinatorial property when acting on a state v_λ, where λ is a partition. Namely, if $n \neq 0$ is an integer and k is a half integer such that $\psi_{k-n}\psi_k^* v_\lambda$ is non-zero, then, up to sign, $\psi_{k-n}\psi_k^* v_\lambda$ will correspond to a partition whose Young diagram is the Young diagram of λ in which we added or removed a border strip of length $|n|$ if $n < 0$ or $n > 0$, respectively. A border strip is a connected (each square shares at least one side with another square) skew diagram with no 2 by 2 squares, where the number of squares is called its length. For example, let $n = -3$ and $S = \{\frac{5}{2}, -\frac{1}{2}, -\frac{5}{2}, -\frac{7}{2}, -\frac{9}{2}, \ldots\}$ as in the previous example. Then v_S, v_1, v_2 and

v_3 in (3.17) correspond to the following Young diagrams (in Russian notation) where the blue squares indicate the added border strip:



$v_S = v_{(3,1)}$ $v_1 = v_{(6,1)}$ $v_2 = v_{(3,2,2)}$ $v_3 = v_{(3,1,1,1,1)}$

The sign of v_i in (3.17) is equal to -1 raised to the power of the height of the added border strip (the height is the number of rows minus 1 in the ordinary non-rotated diagram).

For any operators A and B define the commutator of A and B as $[A, B] := AB - BA$. The operators α_n and α_m satisfy the Heisenberg commutation relation (see Lemma 3.4.1)

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}.$$

For example, let $n = -3$ and $S = \{\frac{5}{2}, -\frac{1}{2}, -\frac{5}{2}, -\frac{7}{2}, -\frac{9}{2}, \dots\}$ as in the previous example and let $m = 3$. It follows from (3.17) that $\alpha_3\alpha_{-3}v_S = \alpha_3 v_1 - \alpha_3 v_2 + \alpha_3 v_3$. Referring to the Maya diagrams of v_1, v_2 and v_3 we have $\alpha_3 v_1 = v_S$, $\alpha_3 v_2 = -v_S$ and $\alpha_3 v_3 = v_S$. Therefore, $\alpha_3\alpha_{-3}v_S = 3v_S$. None of the black beads in the Maya diagram of v_S have a white bead three beads to the right. This implies $\alpha_3 v_S = 0$ and we obtain the result

$$[\alpha_3, \alpha_{-3}]v_S = (\alpha_3\alpha_{-3} - \alpha_{-3}\alpha_3)v_S = 3v_S.$$

Lemma 3.4.1. *For all integers n and m, the operators α_n and α_m satisfy the Heisenberg commutation relation*

$$[\alpha_n, \alpha_m] = n\delta_{n,-m}. \tag{3.18}$$

Proof. Ignoring the trivial case where $n = m$, we consider the following cases:

(1) Suppose $n = 0$ and $m \neq 0$. Then

$$\alpha_n\alpha_m = \left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_k\psi_k^*:\right)\left(\sum_{\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{\ell-m}\psi_\ell^*\right) = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_k^*\psi_{\ell-m}\psi_\ell^* - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_k^*\psi_k\psi_{\ell-m}\psi_\ell^*. \tag{3.19}$$

The relation (3.7a) implies

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_k^*\psi_{\ell-m}\psi_\ell^* = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k(\delta_{\ell-m,k} - \psi_{\ell-m}\psi_k^*)\psi_\ell^*.$$

As $m \neq 0$ the sums

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_{k+m}^* \quad \text{and} \quad \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_{\ell-m}\psi_k^*\psi_\ell^*$$

are well defined, i.e., when they act on a state in $\Lambda^{\frac{\infty}{2}}V$ they converge to a state in $\Lambda^{\frac{\infty}{2}}V$. Therefore,

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_k^*\psi_{\ell-m}\psi_\ell^* = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_{k+m}^* - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_{\ell-m}\psi_k^*\psi_\ell^*.$$

Using the relations (3.7b) and (3.7c) it follows that

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi_{\ell-m}\psi_k^*\psi_\ell^* = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} (-\psi_{\ell-m}\psi_k)(-\psi_\ell^*\psi_k^*) = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{\ell-m}\psi_k\psi_\ell^*\psi_k^*.$$

Applying (3.7a) again implies

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_{\ell-m}\psi_k\psi_\ell^*\psi_k^* = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_{\ell-m}(\delta_{k,\ell}-\psi_\ell^*\psi_k)\psi_k^* = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_{k-m}\psi_k^* - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_{\ell-m}\psi_\ell^*\psi_k\psi_k^*$$

where the second equality follows from

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_{k-m}\psi_k^* \quad \text{and} \quad \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_{\ell-m}\psi_\ell^*\psi_k\psi_k^*$$

being well defined as $m\neq 0$. Therefore,

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_k\psi_k^*\psi_{\ell-m}\psi_\ell^* = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_k\psi_{k+m}^* - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_{k-m}\psi_k^* + \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k>0}}\psi_{\ell-m}\psi_\ell^*\psi_k\psi_k^*. \tag{3.20}$$

The relation (3.7b) implies

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_k^*\psi_k\psi_{\ell-m}\psi_\ell^* = -\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_k^*\psi_{\ell-m}\psi_k\psi_\ell^*.$$

By the relation (3.7a) we have

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_k^*\psi_{\ell-m}\psi_k\psi_\ell^* = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}(\delta_{\ell-m,k}-\psi_{\ell-m}\psi_k^*)\psi_k\psi_\ell^* = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_k\psi_{k+m}^* - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_k^*\psi_k\psi_\ell^*$$

where the second equality follows from

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_k\psi_{k+m}^* \quad \text{and} \quad \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_k^*\psi_k\psi_\ell^*$$

being well defined as $m\neq 0$. Furthermore,

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_k^*\psi_k\psi_\ell^* = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_k^*(\delta_{k,\ell}-\psi_\ell^*\psi_k) = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{k-m}\psi_k^* - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_k^*\psi_\ell^*\psi_k$$

where the second equality follows from

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{k-m}\psi_k^* \quad \text{and} \quad \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_k^*\psi_\ell^*\psi_k$$

being well defined as $m\neq 0$. The relation (3.7c) implies

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_k^*\psi_\ell^*\psi_k = -\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_\ell^*\psi_k^*\psi_k.$$

Therefore,

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_k^*\psi_k\psi_{\ell-m}\psi_\ell^* = -\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_k\psi_{k+m}^* + \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{k-m}\psi_k^* + \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\k<0}}\psi_{\ell-m}\psi_\ell^*\psi_k^*\psi_k. \tag{3.21}$$

It follows from (3.19), (3.20) and (3.21) that

$$\begin{aligned}
\alpha_n\alpha_m &= \Bigg(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi^*_{k+m} + \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_k\psi^*_{k+m}\Bigg) - \Bigg(\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-m}\psi^*_k + \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_{k-m}\psi^*_k\Bigg)\\
&\quad + \Bigg(\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{\ell-m}\psi^*_\ell\psi_k\psi^*_k - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_{\ell-m}\psi^*_\ell\psi^*_k\psi_k\Bigg)\\
&= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k\psi^*_{k+m} - \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k-m}\psi^*_k + \Bigg(\sum_{\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{\ell-m}\psi^*_\ell\Bigg)\Bigg(\sum_{k\in\mathbb{Z}+\frac{1}{2}} :\psi_k\psi^*_k:\Bigg)\\
&= \alpha_m\alpha_n.
\end{aligned}$$

(2) If $m = 0$ and $n \neq 0$ then $[\alpha_n, \alpha_m] = -[\alpha_m, \alpha_n] = 0$ by the first case.

(3) Suppose $n + m \neq 0$ with $n, m \neq 0$. Then

$$\alpha_n\alpha_m = \Bigg(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi^*_k\Bigg)\Bigg(\sum_{\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{\ell-m}\psi^*_\ell\Bigg) = \sum_{k,\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi^*_k\psi_{\ell-m}\psi^*_\ell.$$

The relation (3.7a) implies

$$\alpha_n\alpha_m = \sum_{k,\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n}\big(\delta_{\ell-m,k} - \psi_{\ell-m}\psi^*_k\big)\psi^*_\ell.$$

As $n + m \neq 0$ the sum

$$\sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi^*_{k+m} = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n-m}\psi^*_k = \alpha_{n+m}$$

is well defined. Therefore,

$$\alpha_n\alpha_m = \alpha_{n+m} - \sum_{k,\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi_{\ell-m}\psi^*_k\psi^*_\ell = \alpha_{n+m} - \sum_{k,\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{\ell-m}\psi_{k-n}\psi^*_\ell\psi^*_k \tag{3.22}$$

where the second equality follows from (3.7b) and (3.7c). Again by (3.7a) and by the assumption $n + m \neq 0$ we have

$$\begin{aligned}
\sum_{k,\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{\ell-m}\psi_{k-n}\psi^*_\ell\psi^*_k &= \sum_{k,\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{\ell-m}(\delta_{\ell,k-n} - \psi^*_\ell\psi_{k-n})\psi^*_k\\
&= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n-m}\psi^*_k - \sum_{k,\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{\ell-m}\psi^*_\ell\psi_{k-n}\psi^*_k\\
&= \alpha_{n+m} - \alpha_m\alpha_n.
\end{aligned}\tag{3.23}$$

It follows from (3.22) and (3.23) that

$$\alpha_n\alpha_m = \alpha_{n+m} - (\alpha_{n+m} - \alpha_m\alpha_n) = \alpha_m\alpha_n.$$

(4) Suppose $n + m = 0$ with $n \neq 0$. Note that we could have included the proof of the previous case into the proof of this case. However, the proofs were separated to illustrate the subtle differences that lead to the non-trivial case of (3.18). In particular we must split our first sum into two cases of k to preserve convergence, namely

$$\alpha_n\alpha_m = \sum_{k,\ell\in\mathbb{Z}+\frac{1}{2}} \psi_{k-n}\psi^*_k\psi_{\ell-m}\psi^*_\ell = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_k\psi_{\ell+n}\psi^*_\ell + \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_{k-n}\psi^*_k\psi_{\ell+n}\psi^*_\ell. \tag{3.24}$$

The relation (3.7a) implies

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_k\psi_{\ell+n}\psi^*_\ell = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}(\delta_{\ell+n,k} - \psi_{\ell+n}\psi^*_k)\psi^*_\ell.$$

The sum

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_{k-n}$$

is well defined because when it acts on a state v_S in the basis $\{v_S\}$ of $\Lambda^{\frac{\infty}{2}}V$ it will converge to a state in $\Lambda^{\frac{\infty}{2}}V$ as S_+ is a finite set and the sum is over positive k. Therefore,

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_k\psi_{\ell+n}\psi^*_\ell = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_{k-n} - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi_{\ell+n}\psi^*_k\psi^*_\ell$$

and the relations (3.7b) and (3.7c) imply

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_k\psi_{\ell+n}\psi^*_\ell = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_{k-n} - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{\ell+n}\psi_{k-n}\psi^*_\ell\psi^*_k.$$

As the sum is over positive k and by the relation (3.7a) it follows that

$$\begin{aligned}\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_k\psi_{\ell+n}\psi^*_\ell &= \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_{k-n} - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{\ell+n}(\delta_{k-n,\ell} - \psi^*_\ell\psi_{k-n})\psi^*_k \\ &= \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_{k-n} - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi^*_k + \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{\ell+n}\psi^*_\ell\psi_{k-n}\psi^*_k. \end{aligned} \tag{3.25}$$

Since $n \neq 0$ the relation (3.7a) implies

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_{k-n}\psi^*_k\psi_{\ell+n}\psi^*_\ell = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_{k-n}\psi^*_\ell\psi_{\ell+n}.$$

Also,

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_{k-n}\psi^*_\ell\psi_{\ell+n} = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k(\delta_{k-n,\ell} - \psi^*_\ell\psi_{k-n})\psi_{\ell+n} = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_k - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi^*_\ell\psi_{k-n}\psi_{\ell+n}$$

where the second equality holds as the sums

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_k \quad \text{and} \quad \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi^*_\ell\psi_{k-n}\psi_{\ell+n}$$

are convergent when acting on $\Lambda^{\frac{\infty}{2}}V$ as they sum over negative k. The relations (3.7b) and (3.7c) imply

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_{k-n}\psi^*_\ell\psi_{\ell+n} = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_k - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_\ell\psi^*_k\psi_{\ell+n}\psi_{k-n}.$$

As the sum is over negative k and by the relation (3.7a) it follows that

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_\ell\psi^*_k\psi_{\ell+n}\psi_{k-n} = \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_\ell(\delta_{\ell+n,k} - \psi_{\ell+n}\psi^*_k)\psi_{k-n} = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_{k-n}\psi_{k-n} - \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_\ell\psi_{\ell+n}\psi^*_k\psi_{k-n}.$$

As $n \neq 0$ we have $\psi^*_\ell\psi_{\ell+n} = -\psi_{\ell+n}\psi^*_\ell$ and $\psi^*_k\psi_{k-n} = -\psi_{k-n}\psi^*_k$ by (3.7a). Therefore,

$$\sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_{k-n}\psi^*_k\psi_{\ell+n}\psi^*_\ell = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_k - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_{k-n}\psi_{k-n} + \sum_{\substack{k,\ell\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi_{\ell+n}\psi^*_\ell\psi_{k-n}\psi^*_k. \tag{3.26}$$

It follows from (3.24), (3.25) and (3.26) that

$$\alpha_n\alpha_m = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_{k-n} - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_{k-n}\psi_{k-n} + \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_k - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi^*_k + \alpha_m\alpha_n. \tag{3.27}$$

If $n > 0$ then

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_{k-n} - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi^*_k = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0>k>-n}} \psi_k\psi^*_k, \qquad \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_k - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_{k-n}\psi_{k-n} = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0>k>-n}} \psi^*_k\psi_k$$

and thus by (3.27) and the relation $\psi_k\psi^*_k + \psi^*_k\psi_k = 1$ it follows that

$$\alpha_n\alpha_m = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0>k>-n}} \psi_k\psi^*_k + \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0>k>-n}} \psi^*_k\psi_k + \alpha_m\alpha_n = \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0>k>-n}} (\psi_k\psi^*_k + \psi^*_k\psi_k) + \alpha_m\alpha_n = n + \alpha_m\alpha_n.$$

Otherwise, if $n < 0$ then

$$\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_{k-n}\psi^*_{k-n} - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k>0}} \psi_k\psi^*_k = -\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0<k<m}} \psi_k\psi^*_k, \qquad \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_k\psi_k - \sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ k<0}} \psi^*_{k-n}\psi_{k-n} = -\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0<k<m}} \psi^*_k\psi_k$$

and again by (3.27) and the relation $\psi_k\psi^*_k + \psi^*_k\psi_k = 1$ we have

$$\alpha_n\alpha_m = -\sum_{\substack{k\in\mathbb{Z}+\frac{1}{2}\\ 0<k<m}} (\psi_k\psi^*_k + \psi^*_k\psi_k) + \alpha_m\alpha_n = -m + \alpha_m\alpha_n = n + \alpha_m\alpha_n.$$

In conclusion $\alpha_n\alpha_m = n + \alpha_m\alpha_n$.

In each case of the above cases $[\alpha_n, \alpha_m] = n\delta_{n,-m}$, completing the proof. □

Let α^*_n denote the adjoint operator of α_n. We will use the properties of adjoint operators to show that $\alpha^*_n = \alpha_{-n}$. Let A and B be two operators on a vector space W with adjoint operators A^* and B^*, respectively. Let $(A+B)^*$ denote the adjoint operator of $A+B$. It follows that $\big(f, (A+B)^*g\big) = \big((A+B)f, g\big) = (Af, g) + (Bf, g) = (f, A^*g) + (f, B^*g) = \big(f, (A^* + B^*)g\big)$ for all $f, g \in W$, which implies

$$(A+B)^* = A^* + B^*. \tag{3.28}$$

Let $(AB)^*$ denote the adjoint operator of AB. We have $\big(f, (AB)^*g\big) = \big(ABf, g\big) = (Bf, A^*g) = (f, B^*A^*g)$ for all $f, g \in W$, which implies

$$(AB)^* = B^*A^*. \tag{3.29}$$

It follows from (3.28) and (3.29) that for non-zero n we have

$$\alpha^*_n = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_k\psi^*_{k-n} = \sum_{k\in\mathbb{Z}+\frac{1}{2}} \psi_{k+n}\psi^*_k = \alpha_{-n}$$

and similarly $\alpha^*_0 = \alpha_0$.

We define the generating functions $\psi(z)$ and $\psi^*(z)$ for ψ_k and ψ^*_k, respectively, by

$$\psi(z) = \sum_{i\in\mathbb{Z}+\frac{1}{2}} z^i\psi_i, \qquad \psi^*(z) = \sum_{j\in\mathbb{Z}+\frac{1}{2}} z^{-j}\psi^*_j$$

where z is a formal variable. We have the property

$$\big(\psi(z)f, g\big) = \sum_{i\in\mathbb{Z}+\frac{1}{2}} z^i(\psi_i f, g) = \sum_{i\in\mathbb{Z}+\frac{1}{2}} z^i(f, \psi^*_i g) = \big(f, \psi^*(z^{-1})g\big)$$

for all states $f, g \in \Lambda^{\frac{\infty}{2}}V$. Therefore, we will refer to $\psi^*(z^{-1})$ as the adjoint of $\psi(z)$, even though $\psi(z)$ is not an operator from $\Lambda^{\frac{\infty}{2}}V$ to $\Lambda^{\frac{\infty}{2}}V$.

Lemma 3.4.2. *For all $n \in \mathbb{Z}$ the operator α_n and generating functions $\psi(z)$ and $\psi^*(z)$ satisfy the relations*

$$[\alpha_n, \psi(z)] = z^n \psi(z), \qquad [\alpha_n, \psi^*(z)] = -z^n \psi^*(z).$$

Proof. We have

$$[\alpha_n, \psi(z)] = \sum_{i \in \mathbb{Z}+\frac{1}{2}} z^i [\alpha_n, \psi_i], \qquad [\alpha_n, \psi^*(z)] = \sum_{j \in \mathbb{Z}+\frac{1}{2}} z^{-j} [\alpha_n, \psi_j^*].$$

It follows from (3.7a) that

$$\alpha_n \psi_\ell = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k-n} \psi_k^* \psi_\ell = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k-n} (\delta_{\ell,k} - \psi_\ell \psi_k^*) = \psi_{\ell-n} - \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k-n} \psi_\ell \psi_k^*.$$

The relation (3.7b) implies

$$- \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k-n} \psi_\ell \psi_k^* = \sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_\ell \psi_{k-n} \psi_k^* = \psi_\ell \alpha_n.$$

Therefore, $\psi_{\ell-n} = \alpha_n \psi_\ell - \psi_\ell \alpha_n = [\alpha_n, \psi_\ell]$ and it follows that

$$[\alpha_n, \psi(z)] = \sum_{i \in \mathbb{Z}+\frac{1}{2}} z^i [\alpha_n, \psi_i] = \sum_{i \in \mathbb{Z}+\frac{1}{2}} z^i \psi_{i-n} = \sum_{i \in \mathbb{Z}+\frac{1}{2}} z^{i+n} \psi_i = z^n \psi(z).$$

Taking the adjoint of both sides of the equation $[\alpha_n, \psi(z)] = z^n \psi(z)$ implies

$$[\alpha_{-n}, \psi^*(z^{-1})] = -z^n \psi^*(z^{-1}).$$

Replacing z^{-1} with z we obtain $[\alpha_{-n}, \psi^*(z)] = -z^{-n} \psi^*(z)$. This implies $[\alpha_n, \psi^*(z)] = -z^n \psi^*(z)$. $\square$

For any sequence $\boldsymbol{s} = (s_1, s_2, \ldots)$ of formal variables, we define the vertex operators $\Gamma_\pm$ as

$$\Gamma_\pm(\boldsymbol{s}) = \exp\left(\sum_{n=1}^{\infty} s_n \alpha_{\pm n} \right).$$

We have written $\boldsymbol{s}$ in boldface in $\Gamma_\pm(\boldsymbol{s})$ to signal that it is a sequence of formal variables. We do this to avoid notational ambiguity as we will later define these vertex operators on a single formal variable. We have used shorthand notation in the definition of these vertex operators. We are really working with the formal Taylor series expansion of the exponential function, i.e.,

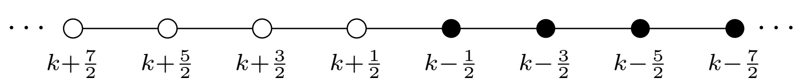
$$\Gamma_\pm(\boldsymbol{s}) = \sum_{m=0}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} s_n \alpha_{\pm n} \right)^m.$$

We will now prove several Lemmas regarding the vertex operators, which will be used in Section 3.5 to give generating functions for plane partitions. The following Lemma shows that $\Gamma_+(\boldsymbol{s})$ acts trivially on the vacuum state $R^k v_0$ in the subspace F^k for all integers k.

Lemma 3.4.3. *Let k be an integer, then*

$$\Gamma_+(\boldsymbol{s}) R^k v_0 = R^k v_0.$$

Proof. The state $R^k v_0$ corresponds to the Maya diagram:



As there are no particles (holes) to the left (right) of k it follows that $\psi_{\ell-n} \psi_\ell^* R^k v_0 = 0$ for all half integers ℓ and positive integers n. Therefore, $\alpha_n R^k v_0 = 0$ for all positive integers n. This implies

$$\left(\sum_{n=1}^{\infty} s_n \alpha_n \right)^m R^k v_0 = 0$$

for all positive integers m. In conclusion

$$\Gamma_+(\boldsymbol{s}) R^k v_0 = R^k v_0 + \sum_{m=1}^{\infty} \frac{1}{m!} \left(\sum_{n=1}^{\infty} s_n \alpha_n \right)^m R^k v_0 = R^k v_0.$$ $\square$

Lemma 3.4.4. *Let $\Gamma_+^*(\boldsymbol{s})$ be the adjoint operator of $\Gamma_+(\boldsymbol{s})$. Then*

$$\Gamma_+^*(\boldsymbol{s}) = \Gamma_-(\boldsymbol{s}).$$

Proof. Let

$$A = \sum_{n=1}^{\infty} s_n \alpha_n, \quad B = \sum_{n=1}^{\infty} s_n \alpha_{-n}$$

and let A^* be the adjoint operator of A. Then by (3.28) and (3.29) we have

$$A^* = \sum_{n=1}^{\infty} s_n \alpha_n^* = \sum_{n=1}^{\infty} s_n \alpha_{-n} = B$$

which implies

$$\Gamma_+^*(\boldsymbol{s}) = \sum_{p=0}^{\infty} \frac{(A^*)^p}{p!} = \sum_{p=0}^{\infty} \frac{B^p}{p!} = \Gamma_-(\boldsymbol{s}).$$

Therefore, $\Gamma_-(\boldsymbol{s})$ is the adjoint operator of $\Gamma_+(\boldsymbol{s})$ and we obtain the result $\Gamma_+^*(\boldsymbol{s}) = \Gamma_-(\boldsymbol{s})$. □

Lemma 3.4.4 can provide a simple way of translating properties about $\Gamma_+(\boldsymbol{s})$ into properties of $\Gamma_-(\boldsymbol{s})$, and vice versa, which will simplify some of the following proofs. The following Lemma will show how the vertex operators Γ_+ and Γ_- commute.

Lemma 3.4.5. *Let $\boldsymbol{s} = (s_1, s_2, \dots)$ and $\boldsymbol{s}' = (s_1', s_2', \cdots)$ be any two sequences of formal variables. Then*

$$\Gamma_+(\boldsymbol{s})\Gamma_-(\boldsymbol{s}') = \exp\left(\sum_{n=1}^{\infty} n s_n s_n'\right)\Gamma_-(\boldsymbol{s}')\Gamma_+(\boldsymbol{s}).$$

Proof. Let

$$A = \sum_{n=1}^{\infty} s_n \alpha_n, \quad B = \sum_{m=1}^{\infty} s_m' \alpha_{-m}, \quad \xi = \sum_{n=1}^{\infty} n s_n s_n'.$$

We will first use induction to show that

$$A^p B = B A^p + p\xi A^{p-1} \tag{3.30}$$

for all non-negative integers p. Clearly the $p = 0$ case holds. The $p = 1$ case of (3.30) holds as

$$AB = \sum_{n,m=1}^{\infty} s_n s_m' \alpha_n \alpha_{-m} = \sum_{n,m=1}^{\infty} s_n s_m' \alpha_{-m} \alpha_n + \sum_{n=1}^{\infty} n s_n s_n' = BA + \xi \tag{3.31}$$

where the second equality follows from Lemma 3.4.1. By this result and induction on p we have

$$\begin{aligned}
A^{p+1}B &= A\big(A^p B\big) \\
&= A\big(BA^p + p\xi A^{p-1}\big) && \text{(by the inductive hypothesis)} \\
&= \big(AB\big)A^p + p\xi A^p \\
&= \big(BA + \xi\big)A^p + p\xi A^p && \text{(by (3.31))} \\
&= BA^{p+1} + (p+1)\xi A^p.
\end{aligned}$$

Therefore, (3.30) holds for all non-negative integers p. We will now use induction to show that

$$A^p B^q = \sum_{k=0}^{\infty} \xi^k k! \binom{p}{k}\binom{q}{k} B^{q-k} A^{p-k} \tag{3.32}$$

for all non-negative integers p and q. Recall that $\binom{p}{k} := 0$ if $k > p$ so that the above sum has finite support. Equation (3.32) clearly holds for all p when $q = 0$. Moreover, (3.32) holds for all p when $q = 1$ by (3.30). It follows by induction on q that

$$\begin{aligned}
A^pB^{q+1} &= (A^pB^q)B \\
&= \left(\sum_{k=0}^{\infty}\xi^k k!\binom{p}{k}\binom{q}{k}B^{q-k}A^{p-k}\right)B && \text{(by the inductive hypothesis)} \\
&= \sum_{k=0}^{\infty}\xi^k k!\binom{p}{k}\binom{q}{k}B^{q-k}(A^{p-k}B) \\
&= \sum_{k=0}^{\infty}\xi^k k!\binom{p}{k}\binom{q}{k}B^{q-k}\big(BA^{p-k}+(p-k)\xi A^{p-k-1}\big) && \text{(by (3.30))} \\
&= \sum_{k=0}^{\infty}\xi^k k!\binom{p}{k}\binom{q}{k}B^{q-k+1}A^{p-k} \\
&\quad + \sum_{k=0}^{\infty}\xi^{k+1}(k+1)!\binom{p}{k+1}\binom{q}{k}B^{q-k}A^{p-k-1} \\
&= \sum_{k=0}^{\infty}\xi^k k!\binom{p}{k}\binom{q}{k}B^{q-k+1}A^{p-k} \\
&\quad + \sum_{k=1}^{\infty}\xi^k k!\binom{p}{k}\binom{q}{k-1}B^{q-k+1}A^{p-k} \\
&= \sum_{k=0}^{\infty}\xi^k k!\binom{p}{k}\binom{q+1}{k}B^{q-k+1}A^{p-k} && \text{(by the identity } \tbinom{q}{k}+\tbinom{q}{k-1}=\tbinom{q+1}{k}\text{)).}
\end{aligned}$$

Therefore, by induction (3.32) holds for all non-negative integers p and q. This implies

$$\sum_{p,q=0}^{\infty}\frac{A^pB^q}{p!q!} = \sum_{k=0}^{\infty}\xi^k k!\sum_{p,q=0}^{\infty}\frac{1}{p!q!}\binom{p}{k}\binom{q}{k}B^{q-k}A^{p-k} = \sum_{k=0}^{\infty}\frac{\xi^k}{k!}\sum_{p,q=0}^{\infty}\frac{B^qA^p}{p!q!}$$

which gives the result

$$\Gamma_+(\boldsymbol{s})\Gamma_-(\boldsymbol{s}') = \sum_{p,q=0}^{\infty}\frac{A^pB^q}{p!q!} = \sum_{k=0}^{\infty}\frac{\xi^k}{k!}\sum_{p,q=0}^{\infty}\frac{B^qA^p}{p!q!} = \exp\left(\sum_{n=1}^{\infty}ns_ns'_n\right)\Gamma_-(\boldsymbol{s}')\Gamma_+(\boldsymbol{s}). \qquad \square$$

Lemma 3.4.5 proves the non-trivial commutation relation for the vertex operators. As $[\alpha_n,\alpha_m]=0$ if both $n,m\le 0$ or both $n,m\ge 0$ it follows immediately that

$$\begin{aligned}
\Gamma_+(\boldsymbol{s})\Gamma_+(\boldsymbol{s}') &= \Gamma_+(\boldsymbol{s}')\Gamma_+(\boldsymbol{s}), \\
\Gamma_-(\boldsymbol{s})\Gamma_-(\boldsymbol{s}') &= \Gamma_-(\boldsymbol{s}')\Gamma_-(\boldsymbol{s})
\end{aligned}$$

for any two sequences $\boldsymbol{s}=(s_1,s_2,\dots)$ and $\boldsymbol{s}'=(s'_1,s'_2,\cdots)$ of formal variables. In the following Lemma we see how the vertex operators commute with the generating functions $\psi(z)$ and $\psi^*(z)$.

Lemma 3.4.6. *Let $\boldsymbol{s}=(s_1,s_2,\dots)$ be any sequence of formal variables. Then*

$$\Gamma_\pm(\boldsymbol{s})\psi(z) = \gamma(z^{\pm1},\boldsymbol{s})\psi(z)\Gamma_\pm(\boldsymbol{s}), \tag{3.33a}$$

$$\Gamma_\pm(\boldsymbol{s})\psi^*(z) = \gamma(z^{\pm1},\boldsymbol{s})^{-1}\psi^*(z)\Gamma_\pm(\boldsymbol{s}) \tag{3.33b}$$

where

$$\gamma(z,\boldsymbol{s}) = \exp\left(\sum_{n=1}^{\infty}s_nz^n\right).$$

Proof. We will first prove (3.33a) and then show how this implies (3.33b). Let

$$\chi_\pm = \sum_{n=1}^{\infty}s_nz^{\pm n}, \quad A_\pm = \sum_{n=1}^{\infty}s_n\alpha_{\pm n}.$$

We will use induction to show that

$$A_\pm^p \psi(z) = \sum_{k=0}^\infty \chi_\pm^k \binom{p}{k} \psi(z) A_\pm^{p-k} \tag{3.34}$$

for all non-negative integers p. Equation (3.34) clearly holds for $p = 0$. The $p = 1$ case of (3.34) holds as

$$A_\pm \psi(z) = \sum_{n=1}^\infty s_n \alpha_{\pm n} \psi(z) = \sum_{n=1}^\infty s_n \psi(z) \alpha_{\pm n} + \sum_{n=1}^\infty s_n z^{\pm n} \psi(z) = \psi(z) A_\pm + \chi_\pm \psi(z) \tag{3.35}$$

where the second equality follows from Lemma 3.4.2. By induction on p we obtain

$$\begin{aligned}
A_\pm^{p+1}\psi(z) &= A_\pm\big(A_\pm^p \psi(z)\big) \\
&= A_\pm \sum_{k=0}^\infty \chi_\pm^k \binom{p}{k} \psi(z) A_\pm^{p-k} && \text{(by the inductive hypothesis)} \\
&= \sum_{k=0}^\infty \chi_\pm^k \binom{p}{k} A_\pm \psi(z) A_\pm^{p-k} \\
&= \sum_{k=0}^\infty \chi_\pm^k \binom{p}{k} \big(\psi(z) A_\pm + \chi_\pm \psi(z)\big) A_\pm^{p-k} && \text{(by (3.35))} \\
&= \sum_{k=0}^\infty \chi_\pm^k \binom{p}{k} \psi(z) A_\pm^{p-k+1} + \sum_{k=0}^\infty \chi_\pm^{k+1} \binom{p}{k} \psi(z) A_\pm^{p-k} \\
&= \sum_{k=0}^\infty \chi_\pm^k \binom{p}{k} \psi(z) A_\pm^{p-k+1} + \sum_{k=1}^\infty \chi_\pm^k \binom{p}{k-1} \psi(z) A_\pm^{p-k+1} \\
&= \sum_{k=0}^\infty \chi_\pm^k \binom{p+1}{k} A_\pm \psi(z) A_\pm^{p-k+1}.
\end{aligned}$$

Therefore, (3.34) holds and implies

$$\Gamma_\pm(\boldsymbol{s})\psi(z) = \sum_{p=0}^\infty \frac{A_\pm^p}{p!}\psi(z) = \sum_{k=0}^\infty \frac{\chi_\pm^k}{k!} \sum_{p=k}^\infty \psi(z) \frac{A_\pm^{p-k}}{(p-k)!} = \sum_{k=0}^\infty \frac{\chi_\pm^k}{k!} \psi(z) \sum_{p=0}^\infty \frac{A_\pm^p}{p!} = \gamma(z^{\pm 1}, \boldsymbol{s})\psi(z)\Gamma_\pm(\boldsymbol{s})$$

which gives the result (3.33a). Recall that the adjoint of $\psi(z)$ is $\psi^*(z^{-1})$. Taking the adjoint of both sides of (3.33a), it follows that

$$\psi^*(z^{-1})\Gamma_\mp(\boldsymbol{s}) = \gamma(z^{\pm 1}, s)\Gamma_\mp(\boldsymbol{s})\psi^*(z^{-1})$$

which gives the result (3.33b). □

3.5 Plane Partitions and Vertex Operators

In this section we will use the results of Section 3.4 to discuss the connection between vertex operators and plane partitions. We have defined the vertex operators $\Gamma_\pm(\boldsymbol{s})$ for arbitrary sequences $\boldsymbol{s} = (s_1, s_2, \ldots)$ of formal variables. Throughout this section we will often want to specialise this sequence by setting $s_n = \frac{x^n}{n}$. Therefore, we will let

$$\Gamma_\pm(x) := \Gamma_\pm\left(\left(\frac{x^n}{n}\right)_{n=1}^\infty\right)$$

where x is a formal variable. We will not have x in boldface to indicate it is a single formal variable and not a sequence of such variables. It follows from Lemma 3.4.5 that

$$\Gamma_+(x)\Gamma_-(y) = \exp\left(\sum_{n=1}^\infty \frac{(xy)^n}{n}\right)\Gamma_-(y)\Gamma_+(x).$$

Moreover, we have the formal power series

$$-\log(1 - xy) = \sum_{n=1}^\infty \frac{(xy)^n}{n}$$

which implies

$$\Gamma_+(x)\Gamma_-(y) = (1-xy)^{-1}\Gamma_-(y)\Gamma_+(x). \tag{3.36}$$

Similarly we also obtain

$$\Gamma_\pm(x)\psi(z) = \exp\left(\sum_{n=1}^\infty \frac{(xz^{\pm1})^n}{n}\right)\psi(z)\Gamma_\pm(x) = (1-xz^{\pm1})^{-1}\psi(z)\Gamma_\pm(x), \tag{3.37a}$$

$$\Gamma_\pm(x)\psi^*(z) = \exp\left(\sum_{n=1}^\infty \frac{(xz^{\pm1})^n}{n}\right)^{-1}\psi^*(z)\Gamma_\pm(x) = (1-xz^{\pm1})\psi^*(z)\Gamma_\pm(x). \tag{3.37b}$$

We will now use (3.15) and (3.37a) to examine how the vertex operators act on the state v_λ where λ is a partition. By (3.37a) we have

$$\begin{aligned}\Gamma_+(x)\psi(z) &= (1-xz)^{-1}\psi(z)\Gamma_+(x)\\ &= \sum_{m=0}^\infty (xz)^m \sum_{i\in\mathbb{Z}+\frac{1}{2}} z^i\psi_i\Gamma_+(x)\\ &= \sum_{i\in\mathbb{Z}+\frac{1}{2}}\sum_{m=0}^\infty x^m z^{m+i}\psi_i\Gamma_+(x)\\ &= \sum_{i\in\mathbb{Z}+\frac{1}{2}} z^i \sum_{m=0}^\infty x^m\psi_{i-m}\Gamma_+(x)\end{aligned}$$

and equating the coefficients of z^i on both sides gives

$$\Gamma_+(x)\psi_i = \sum_{m=0}^\infty x^m\psi_{i-m}\Gamma_+(x).$$

Using this result and (3.15) it follows that

$$\Gamma_+(x)v_\lambda = \left(\prod_{i=1}^n\sum_{m=0}^\infty x^m\psi_{\lambda_i-i-m+\frac{1}{2}}\right)R^{-n}v_0 \tag{3.38}$$

for all integers $n \geq \ell(\lambda)$. Using this result we will prove the following Lemma.

Lemma 3.5.1. *The vertex operator $\Gamma_+(x)$ acts on the state v_λ as follows:*

$$\Gamma_+(x)v_\lambda = \sum_{\mu\prec\lambda} x^{|\lambda-\mu|}v_\mu. \tag{3.39}$$

Proof. For this proof it will be convenient to expand (3.38) into the expression

$$\Gamma_+(x)v_\lambda = \sum_{m_1,\ldots,m_n=0}^\infty x^{m_1+m_2+\cdots+m_n}\psi_{\lambda_1-m_1-\frac{1}{2}}\psi_{\lambda_2-m_2-\frac{3}{2}}\cdots\psi_{\lambda_n-m_n-n+\frac{1}{2}}R^{-n}v_0. \tag{3.40}$$

We will first show that (3.40) can be simplified to a sum over $m_1,\ldots,m_n$ such that $\lambda_i \geq \lambda_i - m_i \geq \lambda_{i+1}$ for each i, that is

$$\Gamma_+(x)v_\lambda = \sum_{\substack{m_1,\ldots,m_n\geq 0\\ \lambda_i\geq\lambda_i-m_i\geq\lambda_{i+1}}} x^{m_1+m_2+\cdots+m_n}\psi_{\lambda_1-m_1-\frac{1}{2}}\psi_{\lambda_2-m_2-\frac{3}{2}}\cdots\psi_{\lambda_n-m_n-n+\frac{1}{2}}R^{-n}v_0. \tag{3.41}$$

To see why we can restrict the m_i's in (3.40) to $\lambda_i - m_i \geq \lambda_{i+1}$ consider the sum

$$\sum_{\substack{m_i,m_{i+1}\geq 0\\ \lambda_i-m_i<\lambda_{i+1}}} x^{m_i+m_{i+1}}\psi_{\lambda_i-m_i-i+\frac{1}{2}}\psi_{\lambda_{i+1}-m_{i+1}-i-\frac{1}{2}}. \tag{3.42}$$

We will now show that this sum is exactly 0. If m_i and m_{i+1} are such that $\lambda_i - m_i = \lambda_{i+1} - m_{i+1} - 1$ then

$$\psi_{\lambda_i - m_i - i + \frac{1}{2}} = \psi_{\lambda_{i+1} - m_{i+1} - i - \frac{1}{2}}$$

and it follows that

$$\psi_{\lambda_i - m_i - i + \frac{1}{2}} \psi_{\lambda_{i+1} - m_{i+1} - i - \frac{1}{2}} = 0.$$

Therefore, (3.42) is equal to the sum

$$\sum_{\substack{m_i, m_{i+1} \geq 0 \\ \lambda_i - m_i < \lambda_{i+1} \\ \lambda_i - m_i \neq \lambda_{i+1} - m_{i+1} - 1}} x^{m_i + m_{i+1}} \psi_{\lambda_i - m_i - i + \frac{1}{2}} \psi_{\lambda_{i+1} - m_{i+1} - i - \frac{1}{2}}. \tag{3.43}$$

For each pair m_i, m_{i+1} satisfying the conditions in the sum (3.43) there exists a unique pair $m_i', m_{i+1}' \geq 0$ such that

$$\begin{aligned} \lambda_i - m_i' &= \lambda_{i+1} - m_{i+1} - 1, \\ \lambda_i - m_i &= \lambda_{i+1} - m_{i+1}' - 1. \end{aligned}$$

These equations, together with the property $\lambda_i - m_i \neq \lambda_{i+1} - m_{i+1} - 1$, imply that

$$\begin{aligned} \lambda_i - m_i' &< \lambda_{i+1}, \\ \lambda_i - m_i' &\neq \lambda_{i+1} - m_{i+1}' - 1, \\ m_i + m_{i+1} &= m_i' + m_{i+1}'. \end{aligned}$$

Hence, the following term

$$x^{m_i' + m_{i+1}'} \psi_{\lambda_i - m_i' - i + \frac{1}{2}} \psi_{\lambda_{i+1} - m_{i+1}' - i - \frac{1}{2}}$$

is in the sum (3.43). As $\lambda_i - m_i' \neq \lambda_{i+1} - m_{i+1}' - 1$ it follows from (3.7a) that

$$\begin{aligned} x^{m_i' + m_{i+1}'} \psi_{\lambda_i - m_i' - i + \frac{1}{2}} \psi_{\lambda_{i+1} - m_{i+1}' - i - \frac{1}{2}} &= -x^{m_i' + m_{i+1}'} \psi_{\lambda_{i+1} - m_{i+1}' - i - \frac{1}{2}} \psi_{\lambda_i - m_i' - i + \frac{1}{2}} \\ &= -x^{m_i + m_{i+1}} \psi_{\lambda_i - m_i - i + \frac{1}{2}} \psi_{\lambda_{i+1} - m_{i+1} - i - \frac{1}{2}}. \end{aligned}$$

Therefore, for every pair m_i, m_{i+1} satisfying the conditions in the sum (3.43) there exists a unique pair m_i', m_{i+1}' also satisfying the conditions in the sum (3.43) such that

$$x^{m_i + m_{i+1}} \psi_{\lambda_i - m_i - i + \frac{1}{2}} \psi_{\lambda_{i+1} - m_{i+1} - i - \frac{1}{2}} + x^{m_i' + m_{i+1}'} \psi_{\lambda_i - m_i' - i + \frac{1}{2}} \psi_{\lambda_{i+1} - m_{i+1}' - i - \frac{1}{2}} = 0.$$

Therefore,

$$\sum_{\substack{m_i, m_{i+1} \\ \lambda_i - m_i < \lambda_{i+1}}} x^{m_i + m_{i+1}} \psi_{\lambda_i - m_i - i + \frac{1}{2}} \psi_{\lambda_{i+1} - m_{i+1} - i - \frac{1}{2}} = 0.$$

This implies

$$\begin{aligned} \Gamma_+(x) v_\lambda &= \sum_{\substack{m_1, \ldots, m_n \geq 0 \\ \lambda_i \geq \lambda_i - m_i \geq \lambda_{i+1}}} x^{m_1 + m_2 + \cdots + m_n} \psi_{\lambda_1 - m_1 - \frac{1}{2}} \psi_{\lambda_2 - m_2 - \frac{3}{2}} \cdots \psi_{\lambda_n - m_n - n + \frac{1}{2}} R^{-n} v_0 \\ &= \sum_{\substack{m_1, \ldots, m_n \geq 0 \\ \lambda_i \geq \lambda_i - m_i \geq \lambda_{i+1}}} x^{\sum_{i=1}^n \lambda_i - (\lambda_i - m_i)} v_{(\lambda_1 - m_1, \lambda_2 - m_2, \ldots)} \\ &= \sum_{\mu \prec \lambda} x^{|\lambda - \mu|} v_\mu. \end{aligned}$$

□

It follows from (3.39) and the relation $(\Gamma_+(x) v_\lambda, v_\mu) = (v_\lambda, \Gamma_-(x) v_\mu)$ that

$$\Gamma_-(x) v_\mu = \sum_{\lambda \succ \mu} x^{|\lambda - \mu|} v_\lambda. \tag{3.44}$$

In the following Theorem we will see how the Schur function arises naturally from vertex operators.

Theorem 3.5.2. *Let $x_1, x_2, \ldots, x_n$ be formal variables, and let λ and μ be partitions such that $\mu \subseteq \lambda$. Then*

$$\big(\Gamma_+(x_1)\cdots\Gamma_+(x_n)v_\lambda, v_\mu\big) = s_{\lambda/\mu}(x_1, \ldots, x_n). \tag{3.45}$$

Proof. When $n = 0$ the equation (3.45) clearly holds as $s_{\lambda/\mu}(\_) = \delta_{\lambda,\mu} = (v_\lambda, v_\mu)$. So we will now assume $n \geq 1$. We will first use induction to show

$$\Gamma_+(x_1)\cdots\Gamma_+(x_n)v_\lambda = \sum_{\substack{\eta^{(0)}\prec\cdots\prec\eta^{(n)}\\ \eta^{(n)}=\lambda}} \left(v_{\eta^{(0)}} \prod_{i=1}^{n} x_i^{|\eta^{(i)}-\eta^{(i-1)}|}\right) \tag{3.46}$$

for all positive integers n. The base case $n = 1$ follows immediately from Lemma 3.5.1. It follows by induction on n that

$$\begin{aligned}
\Gamma_+(x_1)\cdots\Gamma_+(x_{n+1})v_\lambda &= \Gamma_+(x_1)\big(\Gamma_+(x_2)\cdots\Gamma_+(x_{n+1})v_\lambda\big)\\
&= \Gamma_+(x_1)\sum_{\substack{\eta^{(1)}\prec\cdots\prec\eta^{(n+1)}\\ \eta^{(n+1)}=\lambda}} \left(v_{\eta^{(1)}} \prod_{i=2}^{n+1} x_i^{|\eta^{(i)}-\eta^{(i-1)}|}\right) && \text{(by inductive hypothesis)}\\
&= \sum_{\substack{\eta^{(1)}\prec\cdots\prec\eta^{(n+1)}\\ \eta^{(n+1)}=\lambda}} \left(\Gamma_+(x_1)v_{\eta^{(1)}} \prod_{i=2}^{n+1} x_i^{|\eta^{(i)}-\eta^{(i-1)}|}\right)\\
&= \sum_{\substack{\eta^{(0)}\prec\cdots\prec\eta^{(n+1)}\\ \eta^{(n+1)}=\lambda}} \left(v_{\eta^{(0)}} \prod_{i=1}^{n+1} x_i^{|\eta^{(i)}-\eta^{(i-1)}|}\right) && \text{(by (3.39))}
\end{aligned}$$

and thus (3.46) holds. Taking the inner product of both sides of (3.46) with v_μ implies

$$\big(\Gamma_+(x_1)\cdots\Gamma_+(x_n)v_\lambda, v_\mu\big) = \sum_{\substack{\eta^{(0)}\prec\cdots\prec\eta^{(n)}\\ \eta^{(n)}=\lambda\\ \eta^{(0)}=\mu}} \left(\prod_{i=1}^{n} x_i^{|\eta^{(i)}-\eta^{(i-1)}|}\right) = s_{\lambda/\mu}(x_1, \ldots, x_n)$$

where the last equality follows from (1.20). □

Following ideas of Okounkov, Reshetikhin and Vafa we will use vertex operators to derive the generating function for plane partitions [9]. For this we will require an operator that records the size of a partition λ given the state v_λ. We saw such an operator in (3.16). Therefore, we define the *energy operator* H, acting on the zero-charge subspace F_0, as

$$H = \sum_{k\in\mathbb{Z}+\frac{1}{2}} k : \psi_k\psi_k^* :$$

which we know satisfies the property $Hv_\lambda = |\lambda|v_\lambda$. Thus we may think of v_λ as an eigenvector of H with eigenvalue $|\lambda|$, so that $q^H v_\lambda = q^{|\lambda|}v_\lambda$. The operator H is self-adjoint.

Let $\mathcal{B}(r, c, \infty)$ denote the box $\mathcal{B}(r, c, t)$ where t is unbounded. Through the method of diagonal slicing there is a one-to-one correspondence between plane partitions $\pi \subseteq \mathcal{B}(r, c, \infty)$ and sequences of interlacing partitions of the form

$$0 = \lambda^{(-r)} \prec \cdots \prec \lambda^{(-2)} \prec \lambda^{(-1)} \prec \lambda^{(0)} \succ \lambda^{(1)} \succ \lambda^{(2)} \succ \cdots \succ \lambda^{(c)} = 0 \tag{3.47}$$

where 0 denotes the empty partition. The plane partition π corresponding to such a sequence will have

$$|\pi| = \sum_{i=-r}^{c} |\lambda^{(i)}|.$$

Summing over all such sequences of interlacing partitions we obtain the generating function for plane partitions contained in $\mathcal{B}(r, c, \infty)$, namely

$$\sum_{\pi\subseteq\mathcal{B}(r,c,\infty)} q^{|\pi|} = \sum_{\substack{\lambda^{(-r)}\prec\cdots\prec\lambda^{(-1)}\prec\lambda^{(0)}\succ\lambda^{(1)}\succ\cdots\succ\lambda^{(c)}\\ \lambda^{(-r)}=\lambda^{(c)}=0}} \left(\prod_{i=-r}^{c} q^{|\lambda^{(i)}|}\right). \tag{3.48}$$

We will now express this generating function using vertex operators. To achieve this we will use equations (3.39) and (3.44) to generate all the states indexed by all possible interlacing partitions in the sequence (3.47) and during this process we will record the sizes of the partitions in these states using the operator q^H. We begin at the partition $\lambda^{(c)} = 0$ in (3.47) which corresponds to the state v_0. Using $\Gamma_-(1)$ we generate all possible states $v_{\lambda^{(c-1)}}$ over all possible partitions $\lambda^{(c-1)} \succ 0$ of which we then apply q^H to record the sizes of such partitions, namely

$$q^H \Gamma_-(1) v_0 = \sum_{\lambda^{(c-1)} \succ 0} q^{|\lambda^{(c-1)}|} v_{\lambda^{(c-1)}}.$$

Repeatedly applying the operator $q^H \Gamma_-(1)$ a total of c times gives

$$\left(\prod_{i=1}^{c} q^H \Gamma_-(1) \right) v_0 = \sum_{\substack{\lambda^{(0)} \succ \cdots \succ \lambda^{(c)} \\ \lambda^{(c)} = 0}} \left(v_{\lambda^{(0)}} \prod_{i=0}^{c-1} q^{|\lambda^{(i)}|} \right).$$

As $|\lambda^{(c)}| = 0$ we can modify the product in the right-hand side of the above equation and obtain

$$\left(\prod_{i=1}^{c} q^H \Gamma_-(1) \right) v_0 = \sum_{\substack{\lambda^{(0)} \succ \cdots \succ \lambda^{(c)} \\ \lambda^{(c)} = 0}} \left(v_{\lambda^{(0)}} \prod_{i=0}^{c} q^{|\lambda^{(i)}|} \right). \tag{3.49}$$

In much the same way, repeatedly applying the operator $q^H \Gamma_+(1)$ a total of r times to (3.49) gives

$$\left(\prod_{i=1}^{r} q^H \Gamma_+(1) \right) \left(\prod_{i=1}^{c} q^H \Gamma_-(1) \right) v_0 = \sum_{\substack{\lambda^{(-r)} \prec \cdots \prec \lambda^{(-1)} \prec \lambda^{(0)} \succ \lambda^{(1)} \succ \cdots \succ \lambda^{(c)} \\ \lambda^{(c)} = 0}} \left(v_{\lambda^{(-r)}} \prod_{i=-r}^{c} q^{|\lambda^{(i)}|} \right).$$

Taking the inner product with v_0 gives the generating function (3.48), namely

$$\begin{aligned} \left(\left(\prod_{i=1}^{r} q^H \Gamma_+(1) \right) \left(\prod_{i=1}^{c} q^H \Gamma_-(1) \right) v_0, v_0 \right) &= \sum_{\substack{\lambda^{(-r)} \prec \cdots \prec \lambda^{(-1)} \prec \lambda^{(0)} \succ \lambda^{(1)} \succ \cdots \succ \lambda^{(c)} \\ \lambda^{(-r)} = \lambda^{(c)} = 0}} \left(\prod_{i=-r}^{c} q^{|\lambda^{(i)}|} \right) \\ &= \sum_{\pi \subseteq \mathcal{B}(r,c,\infty)} q^{|\pi|}. \end{aligned} \tag{3.50}$$

We will now use this equation to prove Theorem 1.2.2 in the limiting case where $t \to \infty$. For this we first require the following Lemma.

Lemma 3.5.3. *The following relations hold:*

$$\Gamma_\pm(x) q^H = q^H \Gamma_\pm(x q^{\pm 1}).$$

Proof. As H acts on the zero-charge subspace F_0, with basis $\{v_\lambda\}$ the set of states indexed by partitions, it suffices to show

$$\Gamma_\pm(x) q^H v_\lambda = q^H \Gamma_\pm(x q^{\pm 1}) v_\lambda$$

for all partitions λ. For an arbitrary partition λ it follows from (3.39) that

$$\Gamma_+(x) q^H v_\lambda = q^{|\lambda|} \Gamma_+(x) v_\lambda = \sum_{\mu \prec \lambda} x^{|\lambda - \mu|} q^{|\lambda|} v_\mu = \sum_{\mu \prec \lambda} (xq)^{|\lambda - \mu|} q^{|\mu|} v_\mu = q^H \sum_{\mu \prec \lambda} (xq)^{|\lambda - \mu|} v_\mu = q^H \Gamma_+(xq) v_\lambda.$$

Therefore, the relation $\Gamma_+(x) q^H = q^H \Gamma_+(xq)$ holds. Taking the adjoint of both sides of this relation gives

$$q^H \Gamma_-(x) = \Gamma_-(xq) q^H.$$

Replacing the formal variable x with xq^{-1} implies the relation $q^H \Gamma_-(xq^{-1}) = \Gamma_-(x) q^H$. □

It follows from Lemma 3.5.3 and induction that

$$\prod_{i=1}^{r} q^H \Gamma_+(1) = q^{rH}\Big(\prod_{i=1}^{r}\Gamma_+(q^{i-1})\Big), \qquad \prod_{j=1}^{c} q^H \Gamma_-(1) = \Big(\prod_{j=1}^{c}\Gamma_-(q^{j})\Big)q^{cH} \tag{3.51}$$

where $q^{nH} := \prod_{i=1}^{n} q^H$. Applying (3.51) to (3.50) gives

$$\sum_{\pi\subseteq\mathcal{B}(r,c,\infty)} q^{|\pi|} = \left(q^{rH}\Big(\prod_{i=1}^{r}\Gamma_+(q^{i-1})\Big)\Big(\prod_{j=1}^{c}\Gamma_-(q^{j})\Big)q^{cH}v_0, v_0\right).$$

As H is self-adjoint and $q^H v_0 = v_0$ it follows that

$$\begin{aligned}\sum_{\pi\subseteq\mathcal{B}(r,c,\infty)} q^{|\pi|} &= \left(\Big(\prod_{i=1}^{r}\Gamma_+(q^{i-1})\Big)\Big(\prod_{j=1}^{c}\Gamma_-(q^{j})\Big)q^{cH}v_0, q^{rH}v_0\right)\\ &= \left(\Big(\prod_{i=1}^{r}\Gamma_+(q^{i-1})\Big)\Big(\prod_{j=1}^{c}\Gamma_-(q^{j})\Big)v_0, v_0\right).\end{aligned}$$

The relation (3.36) implies

$$\sum_{\pi\subseteq\mathcal{B}(r,c,\infty)} q^{|\pi|} = \left(\prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1}{1-q^{i+j-1}}\right)\left(\Big(\prod_{j=1}^{c}\Gamma_-(q^{j})\Big)\Big(\prod_{i=1}^{r}\Gamma_+(q^{i-1})\Big)v_0, v_0\right).$$

Taking the adjoint of the operators $\Gamma_-(q^j)$ and applying Lemma 3.4.3 gives

$$\begin{aligned}\sum_{\pi\subseteq\mathcal{B}(r,c,\infty)} q^{|\pi|} &= \left(\prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1}{1-q^{i+j-1}}\right)\left(\Big(\prod_{i=1}^{r}\Gamma_+(q^{i-1})\Big)v_0, \Big(\prod_{j=1}^{c}\Gamma_+(q^{j})\Big)v_0\right)\\ &= \left(\prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1}{1-q^{i+j-1}}\right)(v_0, v_0)\\ &= \prod_{i=1}^{r}\prod_{j=1}^{c}\frac{1}{1-q^{i+j-1}}.\end{aligned}$$

This proves Theorem 1.2.2 in the limiting case where $t\to\infty$. Letting both r and c tend to infinity, we obtain the generating function for unbounded plane partitions

$$\sum_{\pi} q^{|\pi|} = \prod_{i=1}^{\infty}\prod_{j=1}^{\infty}\frac{1}{1-q^{i+j-1}} = \prod_{i=1}^{\infty}\frac{1}{(1-q^i)^i}.$$

Summary

Throughout this thesis we used classical through to modern mathematical techniques and results to study generating functions for plane partitions. In Chapter 1 we saw that Schur functions provide a natural generating function for semistandard tableaux. We formed bijections between semistandard tableaux, bounded column strict plane partitions, bounded plane partitions and bounded symmetric plane partitions to give the generating function for each of these as sums over Schur functions. In Chapter 2 we outlined Macdonald's proof of MacMahon's conjecture. Thereafter, in Chapter 3 we discussed vertex operators on the fermionic Fock space which provided modern mathematical methods for studying plane partitions. In particular we used the idea of diagonal slicing to generate all bounded plane partitions by acting on states indexed by partitions using the vertex operators. This provided a way of expressing the generating function for bounded plane partitions in terms of vertex operators. We then manipulated this expression into the generating function given by MacMahon. We were able to achieve this since the operators we defined on the fermionic Fock space satisfied sufficiently many nice relations, with which to perform calculations.

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